

Boundary Moufang trees with abelian root groups of characteristic p

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Abstract

We prove that Moufang sets with abelian root groups arising at infinity of a locally finite tree all come from rank one simple algebraic groups over local fields.

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1 Introduction

A **Moufang set** is a pair $(X, (U_x)_{x \in X})$ consisting of a set X and a collection of permutation groups $U_x \leq \text{Sym}(X)$, called **root groups**, such that for all $x \in X$, the group U_x fixes x , acts regularly on $X \setminus \{x\}$, and preserves the set $\{U_y \mid y \in X\}$ under conjugation. A key example is the set G/P , where G is the group of k -points of a simple algebraic group defined over a field k , and of k -rank one, and P is the group of k -points of a minimal k -parabolic subgroup. Moufang sets have been introduced by J. Tits [Tit92] and F. Timmesfeld [Tim01], in an attempt at axiomatizing the subgroup combinatorics of simple algebraic groups of rank one. An obvious class of Moufang sets is provided by sharply 2-transitive permutation groups. A Moufang set which is not of that form is called **proper**. It is known that every quadratic Jordan division algebra gives rise to a Moufang set with abelian root groups, see [DMW06, Theorem 4.2]. A major conjecture on Moufang sets predicts that, conversely, every proper Moufang set with abelian root groups arises in this way. For a survey on partial results in the direction, we refer to §7.6 in [DMS09]; since then, a major breakthrough was accomplished by Y. Segev in [Seg09]. The goal of the present paper is to prove the conjecture in case the Moufang set is the set of ends of a locally finite tree T , and the root groups are closed subgroups of $\text{Aut}(T)$. That case is relevant both to the theory of Moufang sets, and to the structure theory of locally compact groups.

Let T be a locally finite tree which is **thick**, i.e. all of whose vertices have degree ≥ 3 . Let $(U_\epsilon \mid \epsilon \in \partial T)$ be a **system of root groups** for T , i.e. a collection of subgroups $U_\epsilon \leq \text{Aut}(T)$ such that for all $\epsilon \in \partial T$, the group U_ϵ fixes ϵ , acts regularly on $\partial T \setminus \{\epsilon\}$, and preserves $\{U_f \mid f \in \partial T\}$ under conjugation.

Theorem 1.1. *Assume that the root groups are abelian, and are closed subgroups of $\text{Aut}(T)$ endowed with the compact-open topology.*

Then there exists a non-Archimedean local field k , a simply connected k -simple algebraic group \mathbf{G} defined over k , and a continuous central homomorphism $\varphi: \mathbf{G}(k) \rightarrow \text{Aut}(T)$ such that T is equivariantly isomorphic to the Bruhat–Tits tree of $\mathbf{G}(k)$, and $\varphi(\mathbf{G}(k)) = G^\dagger$, where $G^\dagger = \langle U_\epsilon \mid \epsilon \in \partial T \rangle$ is the little projective group of the given Moufang set. In particular G^\dagger is a closed subgroup of $\text{Aut}(T)$.

We emphasize that the isotropic simple algebraic groups over local fields have been classified by Kneser (in characteristic 0) and Bruhat–Tits (in positive characteristic), see [Tit79]. Following [Tit66] and [Tit79], it turns out that the groups $\mathbf{G}(k)$ satisfying the conditions of Theorem 1.1, namely \mathbf{G} is simply connected, k -simple, of k -rank one and with abelian root groups over k , are precisely the members of following two families:

- $\mathbf{G}(k) \cong \text{SL}_2(D)$, where D is a finite-dimensional central division algebra over k . In that case, in the notation of [Tit79], the Tits-index of $\mathbf{G}(k)$ is ${}^1A_{2d-1,1}^{(d)}$, where $d \geq 1$ is the degree of D . Moreover, the tree T is regular of degree $q + 1$, where q equals the order of the residue field of k .
- $\mathbf{G}(k) \cong \text{SU}_2(D, h)$, where D is the quaternion central division algebra over k , and h is an antihermitian sesquilinear form of Witt index 1 relative to an involution $*$ of the first kind of D such that the space $\{x \in D \mid x^* = x\}$ has dimension 3. In that case the Tits-index of $\mathbf{G}(k)$ is $C_{2,1}^{(2)}$. Moreover, the tree T is semi-regular of degree $(q + 1, q^2 + 1)$, where q equals the order of the residue field of k .

The proof of Theorem 1.1 can be outlined as follows. It is known that if a Moufang set has abelian root groups, then the root group is either torsion-free and uniquely divisible, or of exponent p for some prime p , see [DMS09, Proposition 7.2.2]. In other words, the root group carries the structure of a vector space over a prime field. In characteristic 0, i.e. when the root group is torsion-free, Theorem 1.1 follows from [CDM13, Theorem B]. The latter relies on deep results on p -adic analytic groups due to Lazard–Lubotzky–Mann, that were used to prove that the closure of the subgroup of $\text{Aut}(T)$ generated by the root groups must actually be an algebraic group over \mathbf{Q}_p . The present paper focuses on the case of positive characteristic. The strategy of proof is necessarily different, since no analogue of the aforementioned characterizations of p -adic analytic groups is known to hold in positive characteristic. The main idea behind our proof is the following. Given two distinct points $\mathfrak{e}, \mathfrak{f} \in \partial T$, we construct a countable collection of finite root subgroups $U_m^+ \leq U_{\mathfrak{e}}$ and $U_n^- \leq U_{\mathfrak{f}}$, where $m, n \in \mathbf{Z}$, generating a dense subgroup Γ of the locally compact group $G = \overline{\langle U_{\mathfrak{e}} \cup U_{\mathfrak{f}} \rangle} \leq \text{Aut}(T)$, and forming a so-called **RGD-system**, as defined by J. Tits [Tit92]. This means in particular that the tree T admits a twin tree T' on which Γ acts by automorphisms, and such that the diagonal Γ -action on $T \times T'$ preserves the twinning. The presence of the finite root groups U_m^+ and U_n^- ensures that the twinning satisfies the Moufang condition. At that point, we have obtained a Moufang twin tree, whose set of ends forms a Moufang set with abelian root groups. Those Moufang twin trees have been studied comprehensively by the second author in [Grü14b] (without any hypothesis of local finiteness). The conclusion of Theorem 1.1 can then be deduced from the main result of loc. cit.

It should be pointed out that the group Γ constructed along the proof happens to be a non-uniform lattice in the product group $G \times \text{Aut}(T')$. The fact that our analysis of the class of locally compact groups G under consideration has involved the construction of a lattice Γ in $G \times \text{Aut}(T')$ is an interesting feature: the familiar approach consisting in reducing the study of lattices to the study of their ambient locally compact groups has been reversed here.

We now describe applications of Theorem 1.1. The first provides a criterion allowing one to identify rank one algebraic groups with abelian root subgroups among locally compact groups with a boundary transitive action on a locally finite tree.

Let $G \leq \text{Aut}(T)$ be a closed non-compact subgroup acting transitively on ∂T . Besides rank one algebraic groups over non-Archimedean local fields, there are several other sources of groups $G \leq \text{Aut}(T)$ as above: complete Kac–Moody groups of rank two over finite fields, the full automorphism group $G = \text{Aut}(T)$ in case T is regular or semi-regular, and many variations such as groups with prescribed local actions as defined and studied by Burger–Mozes [BM00].

General results by Burger–Mozes, valid in all cases, imply that the group $G^{(\infty)}$, defined as the intersection of all non-trivial closed normal subgroups of G , is a compactly generated, topologically simple closed subgroup of $\text{Aut}(T)$ acting doubly transitively on ∂T , and such that $G/G^{(\infty)}$ is compact (see [CDM13, Theorem 2.2]).

Corollary 1.2. *Let T be a thick locally finite tree and $G \leq \text{Aut}(T)$ be a closed non-compact group acting transitively on ∂T .*

Then the following assertions are equivalent.

- (i) *There exists a hyperbolic automorphism $h \in G$ such that the **contraction group** $\text{con}(h) = \{g \in G \mid \lim_n h^n g h^{-n} = 1\}$ is abelian.*
- (ii) *There exists a non-Archimedean local field k , a simply connected k -simple algebraic group \mathbf{G} defined over k with abelian root groups over k , and a continuous central sur-*

jective homomorphism $\varphi: \mathbf{G}(k) \rightarrow G^{(\infty)}$. In particular G is isomorphic to a closed subgroup of $\text{Aut}(\mathbf{G}(k))$ containing $\text{Inn}(\mathbf{G}(k)) \cong \mathbf{G}(k)/Z \cong G^{(\infty)}$.

We may combine Corollary 1.2 with general results on the structure of locally compact groups from [CCMT] to derive the following (compare Corollary D from [CDM13]).

Corollary 1.3. *Let $G \neq \{1\}$ be a unimodular locally compact group whose only compact normal subgroup is the trivial one. Assume that for some $h \in G$, the contraction group $\text{con}(h)$ is abelian, and that the closed subgroup $\overline{\langle h\text{con}(h) \rangle}$ is cocompact in G .*

Then there exists a (possibly Archimedean) local field k , a simply connected k -simple algebraic group \mathbf{G} defined over k and a continuous central homomorphism $\varphi: \mathbf{G}(k) \rightarrow G$ such that $\varphi(\mathbf{G}(k))$ is the smallest closed normal subgroup of G . In particular G is isomorphic to a closed subgroup of $\text{Aut}(\mathbf{G}(k))$ containing $\text{Inn}(\mathbf{G}(k))$.

Another application concerns sharply 3-transitive permutation groups. J. Tits [Tit51] proved that if a Lie group G has a continuous sharply 3-transitive action on a topological space X , then $G = \text{PGL}_2(\mathbf{R})$ or $\text{PGL}_2(\mathbf{C})$ and X is the projective line $\mathbb{P}^1(\mathbf{R})$ or $\mathbb{P}^1(\mathbf{C})$. One expects more generally that the only source of sharply 3-transitive actions of a σ -compact locally compact group G on a compact space X is given by the action of a group G with $\text{PSL}_2(k) \leq G \leq \text{PTL}_2(k)$ over $\mathbb{P}^1(k)$, where k is a local field. Partial progress towards this conjecture has recently been accomplished by Carette–Dreesen [CD12], showing that if G is not a Lie group, then G has a continuous proper action by automorphisms on a regular locally finite tree T , such that X is equivariantly homeomorphic to the set of ends ∂T . We are then very close to a situation where Theorem 1.1 applies; we indeed obtain the following.

Corollary 1.4. *Let G be a σ -compact locally compact group and Ω be a compact G -space, such that the G -action on Ω is sharply 3-transitive. Then the following assertions are equivalent.*

- (i) *For $\omega \in \Omega$, there exists a normal subgroup $U_\omega \leq G_\omega$ acting regularly on $\Omega \setminus \{\omega\}$.*
- (ii) *There exists a (possibly Archimedean) local field k and a group $\text{PSL}_2(k) \leq H \leq \text{PTL}_2(k)$ such that (G, Ω) and $(H, \mathbb{P}^1(k))$ are isomorphic topological transformation groups.*

In certain cases, the condition (i) in Corollary 1.4 is known to be automatically satisfied. Indeed, given any (abstract) sharply 3-transitive group G , there is a way to define a **permutational characteristic** of G , which is either 0 or a prime number p . It is known that if the characteristic of G is either 3 or congruent to 1 modulo 3, then the condition (i) in Corollary 1.4 automatically holds (see [Ker74, §13] and Corollary 5.1 below).

2 Preliminaries

2.1 Generalities on Moufang sets

We briefly recall some general conventions and notation from the basic theory of Moufang sets; an excellent exposition of the material can be found in [DMS09].

Given a group H , the set of non-trivial elements of H is denoted by $H^\#$.

Let $(X, (U_x)_{x \in X})$ and pick any point $\infty \in X$. The group $G^\dagger = \langle U_x \mid x \in X \rangle$ is called the **little projective group** of the Moufang set. It is doubly transitive on X . Given two elements $0, \infty \in X$ and any element $\tau \in G^\dagger$ swapping 0 and ∞ , we may identify the set $X \setminus \{\infty\}$ with

the root group U_∞ by sending $x \in X \setminus \{\infty\}$ to the unique element $\alpha_x \in U_\infty$ such that $0\alpha_x = x$. Under this identification, the permutation τ may be viewed as a permutation of the set $(U_\infty)^\#$. It turns out that the whole Moufang set $(X, (U_x)_{x \in X})$ can be recovered from the pair (U_∞, τ) . This Moufang set is denoted by $\mathbb{M}(U_\infty, \tau)$.

For example, if X is the projective line over a field \mathbf{F} and $U_x \leq G = \mathrm{PGL}_2(\mathbf{F})$ is the unipotent radical of the stabiliser G_x , then $(X, (U_x)_{x \in X})$ is a Moufang set. It can be identified with the Moufang set $\mathbb{M}(U, \tau)$, where U is the additive group of \mathbf{F} and $\tau: \mathbf{F}^\# \rightarrow \mathbf{F}^\# : x \mapsto -x^{-1}$. This Moufang set will be denoted by $\mathbb{M}(\mathbf{F})$.

We now come back to a general Moufang set $M(U_\infty, \tau)$. Set $U = U_\infty$. It is customary to denote the group U additively, even though it is not necessarily commutative. For any $a \in U^\#$, there is a unique element μ_a of the double coset $(U^\tau)a(U^\tau)$ which swaps 0 and ∞ . The element $h_a = \tau\mu_a \in G^\dagger$ is called the **Hua map** associated with a . The subgroup \mathfrak{H} of G^\dagger generated by the Hua maps is called the **Hua group**. The Moufang set is **proper** if \mathfrak{H} is non-trivial; this means equivalently that G^\dagger is not sharply 2-transitive on X .

The Moufang set $M(U, \tau)$ is called **special** if $-\alpha\tau = (-a)\tau$ for all $a \in U^\#$. A fundamental result due to Y. Segev [Seg09] ensures that if U is abelian and $M(U, \tau)$ is proper, then it is special.

The following basic identities will be needed.

Lemma 2.1. *Let $\mathbb{M}(U, \tau)$ be a Moufang set and $a \in U^\#$. Then we have*

$$(i) \quad \mu_a^{-1} = \mu_{-a}.$$

$$(ii) \quad \mu_{ah} = \mu_a^h \text{ for all } h \in \mathfrak{H}.$$

$$(iii) \quad \text{If } \mathbb{M}(U, \tau) = \mathbb{M}(U, \tau^{-1}), \text{ then } \mu_{a\tau} = \mu_{-a}^\tau. \text{ In particular } \mu_{a\mu_b} = \mu_{-a}^{\mu_b} \text{ for all } b \in U^\#.$$

$$(iv) \quad \text{If } \mathbb{M}(U, \tau) \text{ is special, then } a\mu_a = -a \text{ and } (-a)\mu_a = a.$$

Proof. See 4.3.1 and 7.1.4 of [DMS09]. □

Lemma 2.2. *Let $\mathbb{M}(U, \tau)$ be a Moufang set and $a, b \in U^\#$ with $a \neq b$. Then the following assertions hold.*

$$(i) \quad (a\tau^{-1} - b\tau^{-1})\tau = (a - b)\mu_b + (-b\tau^{-1})\tau.$$

$$(ii) \quad \mu_{-b}\mu_{b-a}\mu_a = \mu_{(a\tau^{-1} - b\tau^{-1})\tau}.$$

Proof. See 6.1.1 in [DMS09]. □

When the root group U is abelian, it is customary to define h_a to be the constant zero map on U if $a = 0$. In this way, for all $a, b \in U$, we obtain a well-defined endomorphism

$$h_{a,b}: U \rightarrow U : x \mapsto xh_{a+b} - xh_a - xh_b.$$

Notice that $h_{a,b} = h_{b,a}$.

Lemma 2.3. *Let $\mathbb{M}(U, \tau)$ be a special Moufang set with U abelian. Then $a\tau h_{a,b} = -2b$ for all $a \in U^\#$ and all $b \in U$.*

Proof. See 5.1 in [DMW06]. □

Lemma 2.4. *Let $\mathbb{M}(U, \tau)$ be a special Moufang set.*

(i) *If $a, b \in U^\#$ with $\mu_a = \mu_b$, then $b \in \{a, -a\}$.*

(ii) *$\mu_a = \mu_{-a}$ for all $a \in U^\#$ if and only if U is abelian.*

Proof. Part (i) is 4.9(4) from [DMS08b], while Part (ii) is 6.3 from [DMST08]. \square

Lemma 2.5. *Let $\mathbb{M}(U, \tau)$ be a special Moufang set with U abelian and $\tau = \mu_e$ for some $e \in U^\#$. Then for all $a, b, c \in U$ we have the following.*

(i) $h_{a+b,c} - h_{a,c} - h_{b,c} = h_{a,b+c} - h_{a,b} - h_{a,c}$.

(ii) *If $a \neq 0$, then $a\tau h_{a+b,c} = -2c + a\tau h_{b,c}$.*

Proof. (i) The following argument is similar to that used in the proof of 5.10 from [DMS08b]. We have

$$\begin{aligned} h_a + h_b + h_{a,b} + h_c + h_{a+b,c} &= h_{a+b} + h_c + h_{a+b,c} \\ &= h_{a+b+c} \\ &= h_a + h_{b+c} + h_{a,b+c} \\ &= h_a + h_b + h_c + h_{b,c} + h_{a,b+c} \end{aligned}$$

and hence $h_{a+b,c} + h_{a,b} = h_{b,c} + h_{a,b+c}$ as requested.

(ii) Using Part (a) and Lemma 2.3, we obtain

$$\begin{aligned} 0 &= -2(b+c) + 2b + 2c \\ &= a\tau h_{a,b+c} - a\tau h_{a,b} - a\tau h_{a,c} \\ &= a\tau h_{a+b,c} - a\tau h_{a,c} - a\tau h_{b,c} \\ &= a\tau h_{a+b,c} + 2c - a\tau h_{b,c} \end{aligned}$$

and hence $a\tau h_{a+b,c} = -2c + a\tau h_{b,c}$, as requested. \square

Let $\mathbb{M}(U, \tau)$ be a Moufang set. A **root subgroup** is a subgroup $V \leq U$ such that for some $a \in V^\#$, we have $v\mu_a \in V$ for all $v \in V^\#$. In that case $\mathbb{M}(V, \mu_a|_{V^\#})$ is itself a Moufang set, which is regarded as a Moufang subset of $\mathbb{M}(U, \tau)$.

Lemma 2.6. *Let $\mathbb{M}(U, \tau)$ be a special Moufang set with U abelian. Let V be a root subgroup of U such that the corresponding Moufang subset is isomorphic to $\mathbb{M}(\mathbf{F})$, with \mathbf{F} a finite field.*

Then the group $H_0 = \langle \mu_a \mu_b \mid a, b \in V^\# \rangle$ acts faithfully on V . In particular, it is isomorphic to the cyclic group $(\mathbf{F}^)^2$.*

Proof. Let m be the order of $(\mathbf{F}^*)^2$. Then there are elements $a, b \in V^\#$ such that the image of $h = \mu_a \mu_b$ in $H_0/C_{H_0}(V)$ has order m . For an even integer $i \in \mathbf{N}$, we deduce from Lemma 2.1 that

$$h^i = (\mu_a \mu_b)^i = \mu_a h^{-\frac{i}{2}} \mu_a h^{\frac{i}{2}} = \mu_a \mu_{ah^{\frac{i}{2}}},$$

while for an odd integer $i \in \mathbf{N}$, we have

$$h^i = (\mu_a \mu_b)^i = \mu_a h^{-\frac{i-1}{2}} \mu_b h^{\frac{i-1}{2}} = \mu_a \mu_{bh^{\frac{i-1}{2}}}.$$

Now we set $c = ah^{\frac{m}{2}}$ for m even and $c = bh^{\frac{m-1}{2}}$ for m odd. Then $h^m = \mu_a \mu_c \in C_{H_0}(V)$. This implies that $c \in \{a, -a\}$ by Lemma 2.4. Thus $h^m = 1$. Since $|\mathbf{F}^* : (\mathbf{F}^*)^2| \leq 2$, we have $V^\# = aH_0 \cup bH_0$, so that $\mu_c \mu_d = (\mu_a \mu_c)^{-1} \mu_a \mu_d \in \langle h \rangle$ for all $c, d \in V^\#$. It follows that $H_0 = \langle h \rangle$ is cyclic of order m . \square

2.2 The special role of the field of order 9

The following lemma also appears in [BGM14].

Lemma 2.7. *Let \mathbf{F} be a field and $\sigma \in \text{Aut}(\mathbf{F})$ be an automorphism such that $x^{-1}x^\sigma \in \mathbf{K} = \text{Fix}_\sigma(\mathbf{F})$ for all $x \in (\mathbf{F}^*)^2$. Then $\sigma = 1$ or $\mathbf{F} = \mathbf{F}_9$.*

Proof. Set $\tau = \sigma^2$ if $\text{char}(\mathbf{F}) \neq 2$ and $\tau = \sigma$ if $\text{char}(\mathbf{F}) = 2$. For any $x \in \mathbf{F}^*$ we have $(x^{-1}x^\sigma)^\sigma \in \{x^{-1}x^\sigma, -x^{-1}x^\sigma\}$ because σ fixes $(x^{-1}x^\sigma)^2$. Therefore τ fixes $x^{-1}x^\sigma$. Since $x^{-1}x^{\sigma^2} = x^{-1}x^\sigma(x^{-1}x^\sigma)^\sigma$ we conclude that in every case $x^{-1}x^\tau \in \mathbf{L} := \text{Fix}_\tau(\mathbf{F})$ for all $x \in \mathbf{F}^*$. For $x \in \mathbf{F} \setminus \{0, -1\}$ we have

$$x^{-1}x^\tau - (x+1)^{-1}(x+1)^\tau = x^{-1}(x+1)^{-1}(x^\tau - x) \in \mathbf{L}.$$

Moreover, since $x^{-1}x^{\tau^2} = x^{-1}x^\tau(x^{-1}x^\tau)^\tau \in \mathbf{L}$, we have

$$x^{-\tau^{-1}}(x^\tau - x) = (x^{-1}x^{\tau^2} - x^{-1}x^\tau)^{\tau^{-1}} \in \mathbf{L}.$$

If $x \neq x^\tau$, we obtain $x^{-\tau^{-1}}x(x+1) \in \mathbf{L}$ and hence $x^{-1}x^\tau(x+1)^\tau \in \mathbf{L}$. But this implies $x \in \mathbf{L}$, a contradiction. Therefore $\tau = 1$.

We are therefore left with the case $\text{char}(\mathbf{F}) \neq 2$ and $\sigma^2 = 1$. In that case, there is an element $\omega \in \mathbf{F}$ with $\omega^2 \in \mathbf{K}$, $\omega^\sigma = -\omega$ and $\mathbf{F} = \mathbf{K}(\omega)$. For $\lambda \in \mathbf{K}^*$, we obtain

$$c = (\lambda + \omega)^{-2}(\lambda - \omega)^2 = (\lambda + \omega)^{-2+2\sigma} \in \mathbf{K}$$

and hence

$$c(\lambda^2 + 2\lambda\omega + \omega^2) = \lambda^2 - 2\lambda\omega + \lambda^2.$$

Hence $(2 + 2c)\lambda\omega = (1 - c)(\omega^2 + \lambda^2)$. Since the right hand side belongs to \mathbf{K} , this implies $\omega = 0$ (and hence $\mathbf{K} = \mathbf{F}$ and $\sigma = 1$) or $c = -1$. But then $2(\omega^2 + \lambda^2) = 0$ for all $\lambda \in \mathbf{K}$, so $(\mathbf{K}^*)^2 = \{1\}$. Thus if $\sigma \neq 1$, then $\mathbf{K} = \mathbf{F}_3$ and $\mathbf{F} = \mathbf{F}_9$. \square

We emphasize that \mathbf{F}_9 is indeed a genuine exception in Lemma 2.7. Indeed, denoting the non-trivial automorphism of \mathbf{F}_9 by σ , we have $(x^{-1}x^\sigma)^2 = (x^{-1}x^3)^2 = x^4 \in \mathbf{F}_3$ for all $x \in \mathbf{F}_9^*$. The special situation for $\mathbf{F} = \mathbf{F}_9$ can be explained by the fact that $\text{Mat}_2(\mathbf{F}_3)$ contains exactly three subfields of order 9 whose squares normalize each other. In fact, the group $\text{GL}_2(\mathbf{F}_3)$ has a normal subgroup isomorphic to Q_8 and every cyclic subgroup of order 4 of this normal subgroup generates a field of order 9. That fact will be further exploited in Lemma 2.10 below.

We recall from [Grü14a] that a **multiplicative quadratic map** is a map $q: \mathbf{K} \rightarrow \mathbf{L}$ between two unital rings satisfying the following conditions:

- $q(ab) = q(a)q(b)$ for all $a, b \in K$.
- $q(n) = n^2$ for all $n \in \mathbf{Z}$.
- The map $f: K \times K \rightarrow L: (a, b) \mapsto q(a+b) - q(a) - q(b)$ is biadditive.

The following result, due to the second author, will play an important role here.

Theorem 2.8. *Let K, L be commutative fields, and $q: \mathbf{K} \rightarrow \mathbf{L}$ be a multiplicative quadratic map. Then one of the following holds.*

- (i) There exists a pair $\{\phi_1, \phi_2\}$ of field monomorphisms from K to L such that $q(a) = a^{\phi_1} a^{\phi_2}$ for all $a \in K$.
- (ii) There exists a separable quadratic extension M/L and a field monomorphism $\varphi: K \rightarrow M$ such that $q(a) = N_{M/L}(a^\varphi)$ for all $a \in K$.
- (iii) K and L both have characteristic 2, and q is a field monomorphism.

Proof. Follows from Theorem 1.2 in [Grü14a] and [SV00, 1.6.2]. \square

Corollary 2.9 (Corollary 1.3 in [Grü14a]). *Let K, L be commutative fields, and $q: \mathbf{K} \rightarrow \mathbf{L}$ be a multiplicative quadratic map. Then there exists a unique pair $\{\phi_1, \phi_2\}$ of field monomorphisms from K to the algebraic closure \overline{L} such that $q(a) = a^{\phi_1} a^{\phi_2}$ for all $a \in K$.* \square

In the following lemma, we regard \mathbf{F}_9 as a subring of $\text{End}_{\mathbf{F}_3}(\mathbf{F}_9)$ via the natural embedding $x \mapsto (y \mapsto yx)$.

Lemma 2.10. *Let \mathbf{K} be a field and $q: \mathbf{K} \rightarrow \text{End}_{\mathbf{F}_3}(\mathbf{F}_9)$ be a multiplicative quadratic map such that $q(\mathbf{K}^*)$ and $(\mathbf{F}_9^*)^2$ normalize each other. If $q(\mathbf{K})$ is not contained in \mathbf{F}_9 , then \mathbf{K} has order 9 and $q(\mathbf{K})$ generates a field isomorphic to \mathbf{F}_9 .*

Proof. Let σ be the non-trivial Galois automorphism of \mathbf{F}_9 and let $\Gamma = \mathbf{F}_9^* \rtimes \langle \sigma \rangle$, viewed as the subgroup of the unit group of $\text{End}_{\mathbf{F}_3}(\mathbf{F}_9)$. Notice that Γ coincides with the normaliser of \mathbf{F}_9^* in that unit group. Therefore, by assumption $q(\mathbf{K}^*)$ is contained in Γ but not in \mathbf{F}_9^* .

Let now $x \in \mathbf{K}^*$ be an element with $q(x) \notin \mathbf{F}_9^*$ and $i \in \mathbf{F}_9^*$ be an element of order 4. Then $-1 = i^2 = i^{-1}i^3 = i^{-1}i\sigma = i^{-1}i^{q(x)} = [i, q(x)] \in q(\mathbf{K}^*)$. Since $q(\mathbf{K}^*)$ is abelian, we have two possibilities:

- (i) There is an element $\epsilon \in \mathbf{F}_9^*$ of order 8 such that $q(\mathbf{K}^*) = \langle \varphi \rangle$ with $x\varphi = x^3\epsilon$ for all $x \in \mathbf{F}_9$.
- (ii) $q(\mathbf{K}^*) = \langle -1, \sigma \rangle \cong V_4$.

In the first case $(1 + \varphi) \in \text{End}_{\mathbf{F}_3}(\mathbf{F}_9)$ is invertible and has order 8, so the subring \mathbf{L} generated by φ is a field of order 9 and $q: \mathbf{K} \rightarrow \mathbf{L}$ is a multiplicative quadratic map, so either $|\mathbf{K}| = 9$ or $\mathbf{K} \cong \mathbf{F}_{81}$ and $q(x) = x^{10}$ by Theorem 2.8. But in this case $q(\mathbf{K}^*)$ contains an element of order 8, a contradiction.

Suppose that the second case holds. Then $|\mathbf{K}^* : (\mathbf{K}^*)^2| \geq 4$, so \mathbf{K} is an infinite field of characteristic 3. Thus there are $a, b \in \mathbf{K}^*$ with $a^4 \neq b^4$. Set $x = (a - b)^2, y = (2ab)^2$ and $z = (a + b)^2$. Then $x + y = z$ and $q(x) = q(y) = q(z) = 1$, so $f(x, y) = q(z) - q(x) - q(y) = -1$ and hence $q(a^4 - b^4) = q(x - y) = q(x) + q(y) + f(x, -y) = 1 + 1 + 1 = 0$, a contradiction. \square

2.3 Twin trees and RGD-systems

Twin trees were introduced in [RT94] which we recommend as a reference.

Definition 2.11. Let T_+, T_- be two trees such that every vertex has at least 3 neighbours.

- (i) A symmetric function $\delta^*: T_+ \times T_- \cup T_- \times T_+ \rightarrow \mathbf{N}$ is called a **codistance** if for $\epsilon \in \{+, -\}$, $v_\epsilon \in T_\epsilon$ and $v_{-\epsilon} \in T_{-\epsilon}$ the following hold:
 - (a) For every vertex w_ϵ adjacent to v_ϵ one has $\delta^*(w_\epsilon, v_{-\epsilon}) = \delta^*(v_\epsilon, w_\epsilon) \pm 1$.

(b) If $\delta^*(v_\epsilon, v_{-\epsilon}) > 0$, then there is a unique vertex w_ϵ adjacent to v_ϵ with $\delta^*(w_\epsilon, v_{-\epsilon}) = \delta^*(v_\epsilon, v_{-\epsilon}) + 1$.

(ii) If δ^* is a codistance, then (T_+, T_-, δ^*) is called a **twin tree**.

From [RT94] it follows that T_+ and T_- are isomorphic, semi-regular trees. Therefore one also talks of a **twinning** of T_+ . By [RT99] every semi-regular tree admits uncountably many twinings.

Example 2.12. Let k be a field and $K = k(t)$ the rational function field in one indeterminate over k . For $\epsilon = +, -$ there is a unique valuation v_ϵ on K which is trivial restricted to k such that $v_\epsilon(t^\epsilon) = 1$. Let T_ϵ be the Bruhat-Tits tree for $SL_2(K)$ with respect to the valuation v_ϵ . As noted in [RT94] there is a codistance $\delta^* : T_+ \times T_- \cup T_- \times T_+ \rightarrow \mathbf{N}$ and so (T_+, T_-, δ^*) is a twin tree.

An **automorphism** of a twin tree $T = (T_+, T_-, \delta^*)$ is a pair $g = (g_+, g_-) \in \text{Aut}T_+ \times \text{Aut}T_-$ such that $\delta^*(v_+^{g_+}, v_-^{g_-}) = \delta^*(v_+, v_-)$ for all $(v_+, v_-) \in T_+ \times T_-$. Let $\text{Aut}T$ be the group of all automorphisms of T . One easily sees that $g \in \text{Aut}T$ is determined by g_+ (or g_-) and therefore regards $\text{Aut}T$ as a subgroup of $\text{Aut}T_+$ (or $\text{Aut}T_-$). In the example above, the automorphism group of T is $\text{PGL}_2(k[t, t^{-1}]) \rtimes \mathcal{G}$, where \mathcal{G} is the group of field automorphisms of K which stabilize the two valuations v_+ and v_- .

A **twin apartment** is a pair (Σ_+, Σ_-) such that Σ_ϵ is an apartment in T_ϵ and such that for every vertex v_ϵ in Σ_ϵ there is a unique vertex $v_{-\epsilon}$ in $\Sigma_{-\epsilon}$ with $\delta^*(v_\epsilon, v_{-\epsilon}) = 0$ (and therefore $\delta^*(v_\epsilon, w_{-\epsilon})$ goes to infinity as $w_{-\epsilon}$ goes to infinity in one of the two directions of $\Sigma_{-\epsilon}$). A **twin root** is a pair $\alpha = (\alpha_+, \alpha_-)$ such that α_ϵ is a half-apartment in T_ϵ and such that for $(v_+, v_-) \in \alpha_+ \times \alpha_-$ one has $\delta^*(v_+, v_-) = 0$ if and only if v_ϵ is the extremal vertex of α_ϵ for $\epsilon \in \{+, -\}$. We call $(\alpha_+ \setminus \{v_+\}, \alpha_- \setminus \{v_-\})$ the **interior** of α , where v_ϵ is the extremal vertex of α_ϵ . Note that if w is adjacent to v_ϵ for $\epsilon \in \{+, -\}$, then there is a unique twin apartment containing α and w ([RT94]). Two twin roots α and β are called **opposite** if $(\alpha_+ \cup \beta_+, \alpha_- \cup \beta_-)$ is a twin apartment.

For a twin root α one defines U_α as the group of those $g \in \text{Aut}T$ which stabilize every vertex adjacent to a vertex contained in the interior of α . The group U_α is called the **root group** corresponding to α . By [RT94] the group U_α acts freely on the set of twin apartments containing α . We say that U_α is a **full root group** if this action is transitive. The twin tree T has got the **Moufang property** if for every twin root α the root group U_α is full (or equivalently, if there is a twin apartment Σ such that for every twin root α contained in Σ the root group U_α is full). From [RT94] it follows that if T is a Moufang twin tree and α and β are opposite twin roots of T , then $\langle U_\alpha, U_\beta \rangle$ is a rank one group with unipotent subgroups U_α and U_β .

We repeat the definition of a RGD-system. For simplicity, we only consider those of type \tilde{A}_1 , although the concept can be defined for general root systems.

Definition 2.13. Let G be a group, $H \leq G$ and $(U_n^\epsilon)_{n \in \mathbf{Z}, \epsilon \in \{+, -\}}$ be a family of subgroups of G with $H \leq N_G(U_n^\epsilon)$ for all $n \in \mathbf{Z}$ and $\epsilon \in \{0, 1\}$. Then $(G, (U_n^\epsilon)_{n \in \mathbf{Z}, \epsilon \in \{+, -\}}, H)$ is a **RGD-system** of type \tilde{A}_1 if the following axioms hold:

(RGD0) $U_n^\epsilon \neq 1$ for all $n \in \mathbf{Z}$ and $\epsilon \in \{+, -\}$.

(RGD1) $[U_n^\epsilon, U_m^\epsilon] \subseteq U_{n+1}^\epsilon \dots U_{m-1}^\epsilon$ for $n < m \in \mathbf{Z}$ and $\epsilon \in \{+, -\}$.

(RGD2) For $\epsilon \in \{+, -\}$, $n \in \mathbf{Z}$ and $1 \neq a \in U_n^\epsilon$ there is a unique element $\mu_a \in U_n^{-\epsilon} a U_n^{-\epsilon}$ such that $(U_m^\delta)^{\mu_a} = U_{2n-m}^{-\delta}$ for all $m \in \mathbf{Z}$ and $\delta \in \{+, -\}$.

(RGD3) For $\epsilon \in \{+, -\}$ and $n \in \mathbf{Z}$ we have $U_n^\epsilon \cap U_{-n} = 1$, where $U_{-n} = \langle U_m^{-\epsilon} \mid m \in \mathbf{Z}, \epsilon \in \{+, -\} \rangle$.

(RGD4) G is generated by H and $\bigcup_{n \in \mathbf{Z}, \epsilon \in \{+, -\}} U_n^\epsilon$.

If T is a Moufang twin tree, G the subgroup of its automorphism group generated by all root groups, Σ a twin apartment, H the pointwise stabilizer of Σ and $(U_n^\epsilon)_{n \in \mathbf{Z}, \epsilon \in \{+, -\}}$ the family of root groups corresponding to the roots contained in Σ , then $(G, H, (U_n^\epsilon)_{n \in \mathbf{Z}, \epsilon \in \{+, -\}})$ is a RGD-system of type \tilde{A}_1 . On the other hand, every RGD-system of type \tilde{A}_1 gives rise to a Moufang twin tree of degree $(|U_0^+| + 1, |U_1^+| + 1)$, see for example [AB08]. Therefore, these two concepts are equivalent.

Example 2.14. In the example above, there is a natural action of $\mathrm{PSL}_2(k[t, t^{-1}])$ on $T = (T_+, T_-)$. There is a twin apartment Σ such that the family $(U_n^\epsilon)_{n \in \mathbf{Z}, \epsilon \in \{+, -\}}$ of root groups associated to the twin roots contained in Σ is defined by

$$U_n^+ := \left\{ \begin{pmatrix} 1 & 0 \\ at^n & 1 \end{pmatrix} Z; a \in k \right\} \quad \text{and} \quad U_n^- := \left\{ \begin{pmatrix} 1 & at^{-n} \\ 0 & 1 \end{pmatrix} Z; a \in k \right\},$$

where $Z = Z(\mathrm{SL}_2(K))$, while

$$H := \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} Z; a \in k^* \right\}$$

is the pointwise stabilizer of Σ . The root groups are all full, so T is Moufang.

Definition 2.15. A Moufang twin tree is called a **commutative SL_2 -twin tree** if the following two conditions hold:

- (i) If α, β are two opposite twin roots, then $\langle U_\alpha, U_\beta \rangle \cong \mathrm{SL}_2(F)$ or $\mathrm{PSL}_2(F)$ for a commutative field F .
- (ii) If α, β are two prenilpotent roots (i.e. $\alpha_\epsilon \cap \beta_\epsilon$ is a half-apartment for $\epsilon \in \{+, -\}$), then $[U_\alpha, U_\beta] = 1$.

For the corresponding RGD-system this means that every root group is isomorphic to the additive group of a field and the commutator appearing in (RGD1) is in fact trivial. A classification of commutative SL_2 -twin trees can be found in [BGM14].

In [Grü14b] the second author examined the following question: which commutative SL_2 -twin trees induce a Moufang set on the boundary of T_+ (or on the boundary of T_- , which is equivalent). More precisely, let T be a Moufang twin tree and Σ be a twin apartment of T . Then Σ has two ends in ∂T_+ which we denote by ∞ and \mathfrak{o} . For $\epsilon \in \{\infty, \mathfrak{o}\}$ we let U_ϵ denote the closure of the group generated by all root group U_α with α contained in Σ and $\partial\alpha = \epsilon$. Note that U_ϵ is abelian: The first condition of a commutative SL_2 -twin trees implies that U_α is isomorphic to the additive group of a field, the second condition implies that $[U_\alpha, U_\beta] = 1$ if $\partial\alpha = \partial\beta$. Moreover, the group U_ϵ acts regularly on $\partial T_+ \setminus \{\epsilon\}$. Set $G := \langle U_\infty, U_{\mathfrak{o}} \rangle$. We say that T **induces a Moufang set at infinity** if G is a rank one group with unipotent subgroups U_∞ and $U_{\mathfrak{o}}$ or, equivalently, if $\mathbb{M}(T) := (\partial T_+, (U_\infty^g)_{g \in G})$ is a Moufang set.

The main result of [Grü14b] can be summarized as followed.

Theorem 2.16. *Let T be a commutative SL_2 -twin tree which induces a Moufang set at infinity. Then $\mathbb{M}(T)$ is isomorphic to the Moufang set $\mathbb{M}(J)$ associated with one of the quadratic Jordan division algebras J appearing on the following list:*

(i) J is the skewfield of skew Laurent series over a field, i.e.

$$J = K((t))_\theta := \left\{ \sum_{i=n}^{\infty} a_i t^i \mid n \in \mathbf{Z}, a_i \in K \right\},$$

where K is a field, $\theta \in \mathrm{Aut} K$ and t is an indeterminate over K with $ta = a^\theta t$ for all $a \in K$.

(ii) $J = \mathcal{H}(K((t))_\theta, *)$, where $*$ is the involution of $K((t))_\theta$ defined by $t^* = t$ and $a^* = a^\sigma$ for an involutory automorphism σ of K such that $\sigma\theta\sigma = \theta^{-1}$.

(iii) $J = \left\{ \sum_{i=n}^{\infty} a_i t^i \mid n \in \mathbf{Z}, a_{2i} \in E, a_{2i+1} \in F \right\} \subseteq K((t))$, where K is a field of characteristic 2 and E and F are subfields of K containing all squares of K .

(iv) $J = \left\{ \sum_{i=n}^{\infty} a_i t^i \mid n \in \mathbf{Z}, a_{2i} \in E, a_{2i+1} \in F \right\} \subseteq K((t))_\theta$, where E is a subfield of K containing all squares, θ is an involutory automorphism of K and F is the fixed field of θ . This Jordan algebra is contained in $\mathcal{H}(K((t))_\theta, *)$ with $*$ as in (b) for $\sigma = 1$.

Remarks 2.17. (1) If in case (i) or (ii) the order of θ is finite, then the little projective group G^\dagger of the Moufang set is an algebraic group. The groups in (i) are forms of A_n and the groups in (ii) are forms of C_2 (see the description in the introduction above for more details). If the order of θ is infinite, then G^\dagger is a classical group but not an algebraic group. The cases (iii) and (iv) lead to groups of mixed type. They do not occur when the field K is finite.

(2) If T is locally finite, then the field K must be finite, so either case (i) or (ii) holds. In case (i) the automorphism θ must have finite order n , so the center of $K((t))_\theta$ is the local field $F((t^n))$, where $F = K^\theta$ the fixed field of θ . Hence $K((t))_\theta$ is a division algebra of degree n over its center. Since $\mathrm{Aut} K$ is cyclic for a finite field K , we must have $\theta^2 = 1$ and $\sigma \in \{1, \theta\}$ in case (ii). Hence if $\theta \neq 1$, then $K((t))_\theta$ is a quaternion algebra. If $\mathrm{char} K \neq 2$, then $*$ is a non-standard involution.

(3) We remark that if J is any finite-dimensional quadratic Jordan division algebra over a local function field $K((t))$, then the Moufang set $\mathbb{M}(J)$ appears in our list. By [MZ88] a quadratic Jordan division algebra is a division algebra, a Hermitian algebra, a Jordan algebra of Clifford type or an Albert algebra. There are no Albert division algebras over $K((t))$, and since every quadratic form in more than four indeterminates is isotropic over $K((t))$, every quadratic Jordan division algebra of Clifford type is a subalgebra of a quaternion division algebra. Since the Brauer group $\mathrm{Br}(K((t)))$ is isomorphic to \mathbf{Q}/\mathbf{Z} (see [Ser79], XIII, Prop. 6), for every natural number $n \geq 1$ there are exactly $\varphi(n)$ pairwise non-isomorphic division algebras of degree n with center $K((t))$. These are the algebras $L((u))_\theta$ with $[L : K] = n$, $u^n = t$ and $\mathrm{Gal}(L|K) = \langle \theta \rangle$. Since $L((u))_\theta$ has an involution if and only if $\theta^2 = 1$, our list is exhaustive.

3 Boundary Moufang trees

3.1 The setup

Let T be a thick tree, i.e. a simplicial tree all of whose vertices have degree ≥ 3 . For each vertex $x \in V(T)$, we denote by x^\perp the set of vertices adjacent to, but different from, x . The set of ends of T is denoted by ∂T . We let $(U_\epsilon)_{\epsilon \in \partial T}$ be a collection of subgroups of $\text{Aut}(T)$ such that $(\partial T, (U_\epsilon)_{\epsilon \in \partial T})$ is a Moufang set. The Moufang condition implies that *the root groups U_ϵ all consist of elliptic automorphisms of T* , since a hyperbolic element in U_ϵ would be a non-trivial element fixing an end of T different from ϵ , thus violating the condition that the U_ϵ -action on $\partial T \setminus \{\epsilon\}$ is free. The little projective group of this Moufang set is denoted by G . By construction G is a subgroup of $\text{Aut}(T)$.

Throughout, we fix an ordered apartment $\Sigma = (x_n)_{n \in \mathbf{Z}}$ in T . The end of Σ represented by the half-apartment $\{x_n \mid n \leq 0\}$ (resp. $\{x_n \mid n \geq 0\}$) is denoted by ∞ (resp. \mathfrak{o}).

Let $(U, +)$ be a group and $\alpha : U \rightarrow U_\infty$ be an isomorphism. For each $n \in \mathbf{Z}$, we set $U_n = \alpha^{-1}(U_{\infty, x_n})$, where U_{∞, x_n} denotes the stabiliser of x_n in the root group U_∞ . We have $U_{n+1} \leq U_n$ for all n .

We remark that, for any $e \in U^\#$, the given Moufang set $(\partial T, (U_\epsilon)_{\epsilon \in \partial T})$ is isomorphic to $\mathbb{M}(U, \tau)$, where $\tau = \mu_e$. Given $a \in U^\#$, we let

$$\beta_a = \alpha_{a\tau^{-1}}^\tau$$

be the unique element in $U_\mathfrak{o}$ with $(\mathfrak{o})^{\alpha_a} = \infty^{\beta_a}$.

Lemma 3.1. *Let $n, m \in \mathbf{Z}$. If $a \in U_n \setminus U_{n+1}$ and $x \in U_m \setminus U_{m+1}$, then μ_a maps x_m to x_{2n-m} and $x\mu_a \in U_{2n-m} \setminus U_{2n-m+1}$.*

Proof. Let β_a be the unique element of $U_\mathfrak{o}$ with $(\mathfrak{o})^{\alpha_a} = \infty^{\beta_a}$. Then x_n is fixed by β_a since x_n is the unique common vertex common to the three lines (∞, \mathfrak{o}) , $(\infty, \mathfrak{o}^{\alpha_a})$ and $(\mathfrak{o}, \infty^{\beta_a})$. Thus x_n is fixed by $\mu_a = \beta_{a^{-1}}\alpha_a\beta_a$. Since μ_a interchanges ∞ and \mathfrak{o} , it follows that μ_a acts on $\Sigma = (\mathfrak{o}, \infty)$ as the reflection through x_n . Therefore μ_a maps x_m to x_{2n-m} . By the same argument β_x fixes x_m . Since $\beta_x = \alpha_{x\mu_a}^{-1}$ we have

$$\alpha_{x\mu_a} = \beta_x^{\mu_a} \in (U_{\mathfrak{o}, x_m} \setminus U_{\mathfrak{o}, x_{m-1}})^{\mu_a} = U_{\infty, x_{2n-m}} \setminus U_{\infty, x_{2n-m+1}}$$

so that $x\mu_a \in U_{2n-m} \setminus U_{2n-m+1}$ as desired. \square

Let \mathfrak{H} be the Hua group of $\mathbb{M}(U, \tau)$, so $\mathfrak{H} = G_{\mathfrak{o}, \infty}$. Moreover, we set

$$\mathfrak{H}_n = \langle \mu_a \mu_b \mid a, b \in U_n \setminus U_{n+1} \rangle.$$

By Lemma 3.1, the group \mathfrak{H}_n fixes x_n and thus acts trivially on Σ .

Lemma 3.2. *For all $m, n \in \mathbf{Z}$, the subgroups \mathfrak{H}_m and \mathfrak{H}_n normalize each other.*

Proof. We have observed that \mathfrak{H}_n acts trivially on Σ , and thus fixes x_m . Since $\mathfrak{H}_n \leq \mathfrak{H}$ normalizes $U_\mathfrak{o}$ and U_∞ , it follows that \mathfrak{H}_n normalizes the set $\{\mu_a \mid a \in U_m \setminus U_{m+1}\}$. Therefore \mathfrak{H}_n normalizes \mathfrak{H}_m . \square

3.2 The local Moufang sets

An important point to analyze is when the set of neighbours x^\perp of a vertex $x \in V(T)$ carries the structure of a Moufang set that is invariant under the action of the stabiliser G_x . Throughout this subsection, we assume that

$$U_{n+1} \trianglelefteq U_n \quad \text{for all } n \in \mathbf{Z}.$$

We first show that this condition is indeed sufficient to ensure the existence of canonical Moufang sets localised at each vertex.

Lemma 3.3. *Let x be a vertex of T .*

- (i) *For any $\mathfrak{e} \in \partial T$ and $z \in [x, \mathfrak{e}]$ different from x , the group $U_{\mathfrak{e},x}$ acts trivially on z^\perp .*
- (ii) *Given a vertex $y \in x^\perp$ and two ends $\mathfrak{e}, \mathfrak{f} \in \partial T$ such that $y \in [x, \mathfrak{e}] \cap [x, \mathfrak{f}]$, then $U_{\mathfrak{e},x}$ and $U_{\mathfrak{f},x}$ induce the same permutation groups on the set x^\perp . This subgroup of $\text{Sym}(x^\perp)$ is denoted by U_y .*
- (iii) *The pair $(x^\perp, (U_y)_{y \in x^\perp})$ is a Moufang set, called the **local Moufang set** at x .*

Proof. (i) Since $U_{\mathfrak{e},x} \leq U_{\mathfrak{e},z}$ for all $z \in [x, \mathfrak{e}]$, it suffices to prove the statement in the case where z is adjacent to x . The hypothesis that $U_{n+1} \trianglelefteq U_n$ for all $n \in \mathbf{Z}$ implies that $U_{\mathfrak{e},x} \trianglelefteq U_{\mathfrak{e},z}$. Since $U_{\mathfrak{e},z}$ consists of elliptic elements and acts transitively on $\partial T \setminus \{\mathfrak{e}\}$, we infer that $U_{\mathfrak{e},z}$ is transitive on the vertices in z^\perp not belonging to the ray $[z, \mathfrak{e}]$. This implies that $U_{\mathfrak{e},x} \trianglelefteq U_{\mathfrak{e},z}$ acts trivially on z^\perp .

(ii) Let \mathfrak{e}' be an end with $x \in [y, \mathfrak{e}']$. Then there is $g \in U_{\mathfrak{e}'}$ such that $U_{\mathfrak{e}}^g = U_{\mathfrak{f}}$. Since g is elliptic and since $y \in (\mathfrak{e}', \mathfrak{e}) \cap (\mathfrak{e}', \mathfrak{f})$, it follows that g fixes y . From (a), it follows that g induces the identity on x^\perp , so that $U_{\mathfrak{e},x}$ and $U_{\mathfrak{f},x} = U_{\mathfrak{e},x}^g$ indeed induce the same permutation groups on the set x^\perp .

(iii) Let $y, z \in x^\perp$ be distinct and let $\mathfrak{e}, \mathfrak{f}$ be ends with $y \in [x, \mathfrak{e}]$ and $z \in [x, \mathfrak{f}]$. The image of $U_{\mathfrak{e},x}$ in $\text{Sym}(x^\perp)$ is U_y , while the image of $U_{\mathfrak{f},x}$ is U_z . Let $a \in U_{\mathfrak{e},x} \setminus U_{\mathfrak{e},z}$. Then there is $b \in U_{\mathfrak{f}}$ with $U_{\mathfrak{e}}^b = U_{\mathfrak{f}}^a$ and thus $\mathfrak{e}^b = \mathfrak{f}^a$. Since b is elliptic and since $x \in (\mathfrak{e}, \mathfrak{f}) \cap (\mathfrak{e}^b, \mathfrak{f})$, we conclude again that b fixes x . Thus $U_{\mathfrak{f},x}^a = (U_{\mathfrak{f}} \cap G_x)^a = U_{\mathfrak{f}}^a \cap G_x^a = U_{\mathfrak{e}}^b \cap G_x^b = (U_{\mathfrak{e}} \cap G_x)^b = U_{\mathfrak{e},x}^b$ and so $U_y^b = U_z^a$. \square

Lemma 3.4. *Let $k \in \mathbf{Z}$, let $e \in U_k \setminus U_{k+1}$, and set $\tau = \mu_e$. Let also $m > n \in \mathbf{Z}$. Then for any $a \in U_n \setminus U_{n+1}$ and $b \in U_m$, there exists $d \in U_{2k-2n+m+1}$ such that $(b+a)\tau = d + b\mu_a\tau + a\tau$.*

Proof. Let $\rho = \mu_a$ and set $c = \beta_{b+a}\beta_a^{-1}$. Then we have

$$x_{m+1}^{\alpha_{b+a}} = x_{2n-m-1}^{\beta_{b+a}} = x_{2n-m-1}^{c\beta_a},$$

so that

$$x_{m+1}^{\alpha_b\alpha_a\beta_a^{-1}} = x_{2n-m-1}^c.$$

Since $x_{m+1}^{\alpha_b}$ is adjacent to x_m , it is fixed by β_{-a} in view of Lemma 3.3(i), and we deduce

$$x_{2n-m-1}^c = x_{m+1}^{\alpha_b\beta_{-a}\alpha_a\beta_a^{-1}} = x_{m+1}^{\alpha_b\mu_a} = x_{2n-m-1}^{\mu_a^{-1}\alpha_b\mu_a},$$

where the last equality follows from Lemma 3.1. Thus there exists an element $u \in U_{\mathfrak{o}, 2n-m-1}$ with $c = u\alpha_b^{\mu_a}$. Therefore

$$\alpha_{(b+a)\rho}^{\rho^{-1}} = \beta_{b+a} = c\beta_a = u\alpha_b^\rho \alpha_{a\rho}^{\rho^{-1}}.$$

Applying ρ on both sides, we obtain

$$\alpha_{(b+a)\rho} = u^\rho \alpha_b^{\rho^2} \alpha_{a\rho} = u^\rho \alpha_{b\rho^2} \alpha_{a\rho} = u^\rho \alpha_{b\rho^2+a\rho}.$$

Applying $\rho^{-1}\tau$ we get

$$\alpha_{(b+a)\tau} = u^\tau \alpha_{b\rho\tau+a\tau}.$$

Since $u^\tau \in U_{\mathfrak{o}, 2n-m-1}^\tau = U_{\infty, 2k-2n+m+1}$ by Lemma 3.1, there exists $d \in U_{m-2n+1}$ such that $u^\tau = \alpha_d$, and we obtain $(b+a)\tau = d + b\rho\tau + a\tau$, as required. \square

The following result provides important additional information on the local Moufang sets.

Lemma 3.5. *Let $n \in \mathbf{Z}$ and $e \in U_n \setminus U_{n+1}$ and set $\tau = \mu_e$. Let moreover $\overline{U} = U_n/U_{n+1}$ and $\overline{\tau}$ be the permutation of $\overline{U}^\#$ defined by $x + U_{n+1} \mapsto x\tau + U_{n+1}$. Then $M(\overline{U}, \overline{\tau})$ is a Moufang set which is isomorphic to the local Moufang set at x_n . In particular, if the Moufang set $\mathbb{M}(U, \tau)$ is special, then so are all the local Moufang sets.*

Proof. Applying Lemma 3.4 with $k = n$ and $m = n+1$, we see that the map $\overline{\tau}$ is well defined. The first claim follows from Lemma 3.3, while the second follows from the first. \square

3.3 Local action of the Hua group

In the rest of the subsection, we will assume further that U is abelian. In particular the condition $U_{n+1} \trianglelefteq U_n$ trivially holds for all $n \in \mathbf{Z}$. Notice that the Moufang set $\mathbb{M}(U, \tau)$ is necessarily proper, since the Hua group \mathfrak{H} contains hyperbolic elements: indeed, every element of the form $\mu_a \mu_b$, with $a \in U_0 \setminus U_1$ and $b \in U_1 \setminus U_2$, acts as a translation of length 2 on the apartment Σ . The main result from [Seg09] therefore ensures that $\mathbb{M}(U, \tau)$ is special. This fact will be used extensively, without further notice.

Notice that, in the present setting, the quotient group $\mathfrak{H}_n/\mathfrak{H}_n^{(1)}$ is isomorphic to the Hua group of the local Moufang set at x_n^\perp . A key question consists in understanding how the group \mathfrak{H}_n acts on the local Moufang sets at x_{n+1} and x_{n-1} . This question will be answered with the help of the following two technical but crucial results.

Lemma 3.6. *Assume that U is abelian. Let $i \in \{0, 1\}$, let $e \in U_i \setminus U_{i+1}$ and set $\tau = \mu_e$. Given $a \in U_i \setminus U_{i+1}$ and $b \in U_m$ with $m > i$, the following holds for any $x \in U_i^\#$:*

$$xh_{a,b} \equiv -bh_x h_{x\tau, a} \equiv -bh_{x, a\tau} h_a \pmod{U_{m+1}}.$$

Proof. We only prove this for $i = 0$; the proof is similar in case $i = 1$.

We recall that $\mathbb{M}(U, \tau)$ is special. Moreover $\tau = \tau^{-1}$ by Lemma 2.4.

Let $x \in U_0^\#$. By the definition of the Hua maps (see Definition 3.4 in [DMS09]), we have

$$xh_{a+b} = ((x\tau + a + b)\tau - (a + b)\tau)\tau + a + b.$$

Suppose first $x\tau + a \notin U_1$. If $x \in U_n \setminus U_{n+1}$ with $n \geq 0$, then $x\tau, x\tau + a \in U_{-n} \setminus U_{-n+1}$, so by Lemma 3.4

$$(x\tau + a + b)\tau \equiv (x\tau + a)\tau + bh_{x\tau+a}^{-1} \pmod{U_{m+2n+1}}.$$

By Lemma 2.2(a), we have

$$((x\tau + a)\tau - a\tau)\tau = x\tau\mu_a - a = xh_a - a.$$

If $xh_a - a \in U_1$, then $x\tau\mu_a \equiv a \pmod{U_1}$, so $x\tau \equiv a\mu_a \equiv -a \pmod{U_1}$, contradicting the assumption that $x\tau + a \notin U_1$. Thus $((x\tau + a)\tau - a\tau)\tau \in U_0 \setminus U_1$, so that $(x\tau + a)\tau - a\tau \in U_0 \setminus U_1$ by Lemma 3.1. Now we have

$$\begin{aligned} (x\tau + a + b)\tau - (a + b)\tau &\equiv (x\tau + a)\tau + bh_{x\tau+a}^{-1} - (a + b)\tau \\ &\equiv (x\tau + a)\tau - a\tau + bh_{x\tau+a}^{-1} - bh_a^{-1} \pmod{U_{m+1}}, \end{aligned}$$

where Lemma 3.4 has been used to evaluate $(a + b)\tau$. Applying τ and using Lemma 3.4 once more, we obtain

$$\begin{aligned} ((x\tau + a + b)\tau - (a + b)\tau)\tau &\equiv ((x\tau + a)\tau - a\tau + bh_{x\tau+a}^{-1} - bh_a^{-1})\tau \\ &\equiv ((x\tau + a)\tau - a\tau)\tau + bh_{x\tau+a}^{-1}h_{(x\tau+a)\tau-a\tau}^{-1} - bh_a^{-1}h_{(x\tau+a)\tau-a\tau}^{-1} \end{aligned}$$

modulo U_{m+1} . Applying Lemma 2.2(b), we deduce

$$\begin{aligned} h_{x\tau+a}^{-1}h_{(x\tau+a)\tau-a\tau}^{-1} &= \mu_{x\tau+a}\tau\mu_{(x\tau+a)\tau-a\tau}\tau = \mu_{x\tau+a}\mu_{((x\tau+a)\tau-a\tau)\tau} \\ &= \mu_{x\tau+a}\mu_{x\tau+a}\mu_{x\tau}\mu_a = \tau\mu_x\tau\mu_a \\ &= h_xh_a \end{aligned}$$

and

$$\begin{aligned} h_a^{-1}h_{(x\tau+a)\tau-a\tau}^{-1} &= \mu_a\tau\mu_{(x\tau+a)\tau-a\tau}\tau = \mu_a\mu_{((x\tau+a)\tau-a\tau)\tau} \\ &= \mu_a\mu_{-a}\mu_{-x\tau}\mu_{x\tau+a} = \mu_{x\tau}\mu_{x\tau+a} = \tau\mu_x\tau\mu_{x\tau+a} \\ &= h_xh_{x\tau+a}. \end{aligned}$$

Moreover we have $h_{x\tau} = \tau\mu_{x\tau} = \tau\tau^{-1}\mu_{-x\tau} = h_x^{-1}$. Thus

$$\begin{aligned} xh_{a+b} &= ((x\tau + a + b)\tau - (a + b)\tau)\tau + a + b \\ &\equiv ((x\tau + a)\tau - a\tau)\tau + a + bh_xh_a - bh_xh_{x\tau+a} + b \\ &\equiv xh_a + bh_xh_a - bh_xh_{x\tau+a} + bh_xh_{x\tau} \\ &\equiv xh_a - bh_x(h_{x\tau+a} - h_a - h_{x\tau}) \\ &\equiv xh_a - bh_xh_{x\tau,a} \pmod{U_{m+1}}. \end{aligned}$$

Since $xh_b \in U_{2m} \leq U_{m+1}$, we have

$$xh_{a,b} = xh_{a+b} - xh_a - xh_b \equiv -bh_xh_{x\tau,a} \pmod{U_{m+1}},$$

which confirms the first equality asserted by the lemma. Proposition 5.8(2) from [DMS08b] ensures that $h_xh_{x,a\tau}h_a^{-1} = h_xh_{x\tau,a}h_{a\tau} = h_{x,a\tau}$, and the second equality from the lemma follows.

We now suppose that $x\tau + a \in U_1$. Then Lemma 3.4 ensures

$$x = x\tau^2 = (-a + x\tau + a)\tau \equiv (-a)\tau + (x\tau + a)h_a^{-1} \pmod{U_1},$$

so that $c := x + a\tau = x - (-a)\tau \in U_1$. Therefore

$$xh_{a,b} = (-a\tau + c)h_{a,b} = 2b + ch_{a,b}$$

by Lemma 2.3, and

$$bh_{a\tau,x}h_a = bh_{a\tau,c-a\tau}h_a.$$

Now Lemma 2.5 implies

$$h_{a\tau,c-a\tau} = h_{0,c} - h_{a\tau,c} - h_{-a\tau,c} + h_{a\tau,c} + h_{a\tau,-a\tau} = -h_{-a\tau,c} - 2h_{a\tau} = -h_{-a\tau,c} - 2h_a^{-1}.$$

Since $-c \in U_1$, we have $-c\tau \in U_{-k} \setminus U_{-k+1}$ for some $k > 0$ by Lemma 3.1, so that $a - c\tau \notin U_1$. We may therefore apply the first part of the proof to $-c$ and deduce that $-ch_{a,b} \equiv -bh_{-c,a\tau}h_a \pmod{U_{m+1}}$. Since $\mu_d = \mu_{-d}$ for all $d \in U^\#$ by Lemma 2.4, we also have $h_d = h_{-d}$ and $h_{d,f} = h_{-d,-f}$. Thus we get

$$\begin{aligned} bh_{a\tau,x}h_a &= -bh_{-a\tau,c}h_a - 2b = -bh_{a\tau,-c}h_a - 2b \\ &\equiv -ch_{a,b} - 2b \\ &\equiv -xh_{a,b} \pmod{U_{m+1}}. \end{aligned}$$

As in the first case, this also implies that $xh_{a,b} \equiv -bh_xh_{x\tau,a} \pmod{U_{m+1}}$. \square

Proposition 3.7. *Assume that U is abelian. Let $i \in \{0, 1\}$, let $e \in U_i \setminus U_{i+1}$ and set $\tau = \mu_e$. For any $a \in U_i \setminus U_{i+1}$ and $b \in U_m$ with $m > i$, we have the following:*

- (i) $xh_a \equiv xh_{a+b} \pmod{U_m}$ for all $x \in U_i$.
- (ii) $xh_a \equiv xh_{a+b} \pmod{U_{m+1}}$ for all $x \in U_m$.

Proof. (i) Let $x \in U_i^\#$. By Lemma 3.6, we have $xh_{a+b} - xh_a = xh_{a,b} + xh_b \equiv -bh_{x,a\tau}h_a + xh_b \pmod{U_{m+1}}$. Since xh_b and $-bh_{x,a\tau}h_a$ are both contained in U_m , we deduce that $xh_{a+b} \equiv xh_a \pmod{U_m}$ as desired.

(ii) Let $x \in U_m$ and $y \in U_i \setminus U_{i+1}$. Lemma 3.6 implies that

$$yh_{a,b+x} \equiv yh_{a,b} + yh_{a,x} \pmod{U_{m+1}}.$$

Moreover $yh_{b,x} = yh_{b+x} - yh_b - yh_x \in U_{2m} \leq U_{m+1}$. Therefore, Lemma 2.5 implies

$$\begin{aligned} yh_{a+b,x} &= yh_{a,b+x} - yh_{a,b} + yh_{b,x} \\ &\equiv yh_{a,x} \pmod{U_{m+1}}. \end{aligned}$$

Applying Lemma 3.6 again twice, we obtain

$$-xh_yh_{y\tau,a} \equiv yh_{a,x} \equiv yh_{a+b,x} \equiv -xh_yh_{y\tau,a+b} \pmod{U_{m+1}}.$$

Lemma 3.1 ensures that h_y acts trivially on the apartment Σ , and therefore normalises U_m . We may thus replace x by xh_y^{-1} in the above equation, which yields

$$xh_{y\tau,a} \equiv xh_{y\tau,a+b} \pmod{U_{m+1}}.$$

Replacing $y\tau$ by y , we obtain $xh_{y,a} \equiv xh_{y,a+b} \pmod{U_{m+1}}$. Specialising to $y = -a$, we get $-2xh_a = xh_{a,-a} \equiv xh_{-a,a+b} \equiv xh_b - xh_{-a} - xh_{a+b} \pmod{U_{m+1}}$ and thus $-xh_a \equiv xh_b - xh_{a+b} \equiv -xh_{a+b} \pmod{U_{m+1}}$ since $xh_b \in U_{3m} \leq U_{m+1}$. \square

4 Locally finite abelian boundary Moufang trees

We shall now focus on the case when T is locally finite and U is abelian.

4.1 The local Moufang sets are finite projective lines

Lemma 4.1. *Assume that T is locally finite and that U is abelian. Then for every vertex x , there is a finite field \mathbf{F} such that the local Moufang set at x is isomorphic to $\mathbb{M}(\mathbf{F})$.*

Proof. As observed above, the Moufang set $\mathbb{M}(U, \tau)$ is special as a consequence of [Seg09]. By Lemma 3.5 the local Moufang sets are also special, so the claim follows since every finite special Moufang set is of the form $\mathbb{M}(\mathbf{F})$ for some finite field \mathbf{F} , in view of [Seg08] and [DMS08a]. \square

Recall that the little projective group G is 2-transitive on the set of ends ∂T . This implies that G is edge-transitive on T (see Lemma 3.1.1 from [BM00]). Moreover G is generated by elliptic automorphisms of T , so that G preserves the canonical bipartition of the vertex set. We conclude that G acts with exactly two orbits of vertices. From Lemma 4.1, it follows that there are prime powers q_0 and q_1 such that the local Moufang sets are all isomorphic to $\mathbb{M}(\mathbf{F}_{q_0})$ or $\mathbb{M}(\mathbf{F}_{q_1})$.

4.2 Construction of finite root subgroups

We will now assume further that U is of exponent p for some prime p . This implies that the root groups of the local Moufang sets are also groups of exponent p . Therefore the finite fields \mathbf{F}_{q_0} and \mathbf{F}_{q_1} are both of characteristic p in that case.

Our next goal is to construct a finite root subgroup $V_n \leq U$ fixing the vertex x_n and whose image in $\text{Sym}(x_n^\perp)$ is a root group of the local Moufang set at x_n . This will be achieved in Proposition 4.6 below. We first need to collect a number of technical preparations.

Lemma 4.2. *Assume that T is locally finite and that U is abelian of exponent p for some prime p . Let $i \in \{0, 1\}$ and $n \geq i$. Let $V \leq U_i$ be a subgroup satisfying the following conditions:*

- (1) $U_i = V + U_{i+1}$ if $n > i$, and $V = U_i$ if $n = i$.
- (2) $V \cap U_{i+1} = U_{n+1}$.
- (3) V is normalized by $\tilde{\mathcal{H}} := \langle \mu_a \mu_b \mid a, b \in V^* \rangle$, where $V^* = V \setminus U_{n+1}$.

Let $\tilde{\mathcal{H}}_0 = C_{\tilde{\mathcal{H}}}(V/U_{n+1})$, let $\mathcal{H} = \tilde{\mathcal{H}}/C_{\tilde{\mathcal{H}}}(V/U_{n+2})$ and let \mathcal{H}_0 be the image of $\tilde{\mathcal{H}}_0$ in \mathcal{H} . Then the following assertions hold.

- (i) $[\tilde{\mathcal{H}}_0, U_{n+1}] \leq U_{n+2}$.
- (ii) \mathcal{H}_0 is a normal elementary-abelian p -subgroup of \mathcal{H} .
- (iii) $\mathcal{H}/\mathcal{H}_0$ is cyclic of order $q_i - 1$ if $p = 2$ (resp. $\frac{q_i - 1}{2}$ if $p \neq 2$).
- (iv) For all $e \in V^*$ there is $a \in V^*$ such that the image of $\langle \mu_e \mu_a \rangle$ in \mathcal{H} is a complement for \mathcal{H}_0 .

Proof. (i) Set $N_1 = \tilde{\mathcal{H}}_0 = C_{\tilde{\mathcal{H}}}(V/U_{n+1})$ and $N_2 = C_{\tilde{\mathcal{H}}}(U_{n+1}/U_{n+2})$. Let $e \in V^*$. For $a \in V^*$ set $h_a = \mu_e \mu_a$. Note that $\tilde{\mathcal{H}}$ is generated by $\{h_a; a \in V^*\}$ since $\mu_a \mu_b = h_a^{-1} h_b$ for all $a, b \in V^*$. By Proposition 3.7 we have $h_a^{-1} h_{a+b} \in N_1 \cap N_2$ for all $a \in V^*$ and $b \in U_{n+1}$. In particular, given a subset $X \subset V$ such that $\{0\} \cup X$ is a system of coset representatives for U_{n+1} in V , then the quotient $\tilde{\mathcal{H}}/N_1 \cap N_2$ is generated by the images of $\{h_x \mid x \in X\}$.

By hypothesis, the map $\varphi : V/U_{n+1} \rightarrow U_i/U_{i+1} : a + U_{n+1} \mapsto a + U_{i+1}$ is an isomorphism which is \mathcal{H} -equivariant. It follows that $\tilde{\mathcal{H}}/N_1$ is cyclic of order $q_i - 1$ if $p = 2$ (resp. $\frac{q_i-1}{2}$ if p is odd). Thus there is $a \in V^*$ such that $\tilde{\mathcal{H}}/N_1$ is generated by $h_a N_1$. For m even, we have

$$\begin{aligned} h_a^m &= (\mu_e \mu_a)^m = \underbrace{\mu_e \mu_a \mu_e \dots \mu_a}_{m-1} \underbrace{\mu_e \mu_a \mu_e \dots \mu_a}_{m-1} = \underbrace{\mu_e \mu_a \mu_e \dots \mu_a}_{m-1} \underbrace{\mu_e \mu_e \mu_e}_{m-1} \underbrace{\mu_a \mu_e \dots \mu_a}_{m-1} \\ &= \mu_e h_a^{-\frac{m}{2}} \mu_e h_a^{\frac{m}{2}} = \mu_e \mu_{eh_a^{\frac{m}{2}}} = h_{eh_a^{\frac{m}{2}}}. \end{aligned}$$

Similarly, for m odd, we have

$$\begin{aligned} h_a^m &= (\mu_e \mu_a)^m = \underbrace{\mu_e \mu_a \mu_e \dots \mu_e}_{m-1} \underbrace{\mu_a \mu_e \dots \mu_a}_{m-1} \\ &= \mu_e h_a^{-\frac{m-1}{2}} \mu_a h_a^{\frac{m-1}{2}} = \mu_e \mu_{ah_a^{\frac{m-1}{2}}} = h_{ah_a^{\frac{m-1}{2}}}. \end{aligned}$$

From Lemma 4.1, it follows that $U_i^*/U_{i+1} = (a + U_{i+1})\tilde{\mathcal{H}} \cup (e + U_{i+1})\tilde{\mathcal{H}}$. Transforming by φ , we deduce that $V^*/U_{n+1} = (a + U_{n+1})\tilde{\mathcal{H}} \cup (e + U_{n+1})\tilde{\mathcal{H}}$. Now we set $m = |\tilde{\mathcal{H}}/N_1|$ (so $m = q_i - 1$ if $p = 2$ and $m = \frac{q_i-1}{2}$ if p is odd) and $X = \{eh_a^j; j = 0, \dots, m-1\} \cup \{ah_a^j; j = 0, \dots, m-1\}$. Hence $X \cup \{0\}$ is a system of coset representatives for U_{n+1} in V . We have seen that $\tilde{\mathcal{H}}/N_1 \cap N_2$ is generated by the images of $\{h_x \mid x \in X\}$, and we deduce that $\tilde{\mathcal{H}}/N_1 \cap N_2$ is cyclic and generated by h_a . Moreover, since $h_a^m \in N_1$, by 2.4 there is $b \in U_{n+1}$ such that $eh_a^{\frac{m}{2}} \in \{e + b, -e + b\}$ for m even and $ah_a^{\frac{m-1}{2}} \in \{e + b, -e + b\}$ for m odd. Thus $h_a^m = h_{e+b}$ or $h_a^m = h_{-e+b}$, and we deduce from Proposition 3.7 that $h_a^m \in N_2$. Therefore, we have $N_1 \leq N_2$. This proves (i).

(ii) Clearly $\tilde{\mathcal{H}}_0$ is normal in $\tilde{\mathcal{H}}$, so that \mathcal{H}_0 is normal in \mathcal{H} . Moreover, since \mathcal{H}_0 centralizes U_{n+1}/U_{n+2} by Part (i), the commutator map

$$V/U_{n+1} \times \mathcal{H}_0 \rightarrow U_{n+1}/U_{n+2} : (a + U_{n+1}, h) \mapsto [a, h] + U_{n+2}$$

is a homomorphism. Since the centraliser $C_{\mathcal{H}_0}(V/U_{n+2})$ is trivial, this induces an injective homomorphism of \mathcal{H}_0 into the additive group of homomorphisms from V/U_{n+1} to U_{n+1}/U_{n+2} . It follows \mathcal{H}_0 is an elementary-abelian p -group.

(iii) With the notation of Part (i), we have $\mathcal{H}/\mathcal{H}_0 \cong \tilde{\mathcal{H}}/N_1$, which is cyclic of order $q_i - 1$ for $p = 2$ and $\frac{q_i-1}{2}$ if p is even.

(iv) We have $h_a^m \in N_1 \cap N_2$, thus the image of h_a^m in \mathcal{H} lies in \mathcal{H}_0 . Hence $h_a^m C_{\tilde{\mathcal{H}}}(V/U_{n+2})$ has order 1 or p by (ii). In the latter case we replace a by eh_a if $p = 2$, and by $ah_a^{\frac{p-1}{2}}$ if p is odd. Now $h_a C_{\tilde{\mathcal{H}}}(V/U_{n+2})$ has order m and hence it therefore generates a complement for \mathcal{H}_0 in \mathcal{H} . \square

It is important to control the local action of the subgroup $\tilde{\mathcal{H}} \leq \mathfrak{H}$ afforded by Lemma 4.2 on the local Moufang sets at x_n and x_{n+1} . This is achieved by the next lemma, using a coordinatization of those local Moufang sets by the finite fields \mathbf{F}_{q_0} and \mathbf{F}_{q_1} .

Lemma 4.3. *Retain the notation and hypotheses of Lemma 4.2. Set $\mathbf{F} = V/U_{n+1}$ and $\mathbf{E} = U_{n+1}/U_{n+2}$. Fix $e \in V^*$, and for each $a \in V^\#$, set $h_a = \mu_e \mu_a$. Let also φ_a (resp. ψ_a) be the image of h_a in $\text{End}(\mathbf{F})$ (resp. $\text{End}(\mathbf{E})$), and define φ_0 (resp. ψ_0) as the zero-map of \mathbf{F} (resp. \mathbf{E}). Then we have the following.*

- (i) φ_a and ψ_a depend only on the coset $a + U_{n+1}$. We may thus view φ_a and ψ_a as functions of $a \in \mathbf{F}$.
- (ii) There is a multiplication \cdot on \mathbf{F} such that $(\mathbf{F}, +, \cdot)$ is a finite field with neutral element $e + U_{n+1}$, $a\mu_e + U_{n+1} = -(a + U_{n+1})^{-1}$ for all $a \in V^*$ and $x\varphi_y = x \cdot y^2$ for all $x, y \in \mathbf{F}$. Moreover, $\varphi_{x \cdot y} = \varphi_x \varphi_y$ and $\psi_{x \cdot y} = \psi_x \psi_y$.
- (iii) The map $\psi: \mathbf{F} \rightarrow \text{End}(\mathbf{E}): a \mapsto \psi_a$ is a multiplicative quadratic map.
- (iv) There is a multiplication \cdot on \mathbf{E} such that $(\mathbf{E}, +, \cdot)$ is a finite field and such that there is an embedding ι from \mathbf{F} to the algebraic closure of \mathbf{E} and a natural number $0 \leq s \leq \frac{r_i}{2}$ with $x\psi_y = x \cdot \iota(y)^{1+p^s}$ for all $x \in \mathbf{E}$ and $y \in \mathbf{F}$, where $r_i = \log_p q_i$.

Proof. (i) If $e \in V^*$, then Proposition 3.7 ensures that $\varphi_a = \varphi_{a+b}$ for any $b \in U_{n+1}$. By Lemma 4.2(i), this also implies that $\psi_a = \psi_{a+b}$. If $a \in U_{n+1}^\#$, then $U_m h_a \leq U_{m+2n+2-2i}$ for all $m \in \mathbf{Z}$, so $\varphi_a = 0$ and $\psi_a = 0$.

(ii) Since the local Moufang set at x_i is isomorphic to $\mathbb{M}(\mathbf{F}_{q_i})$, we have a multiplication \cdot on U_i/U_{i+1} with $(a + U_{i+1}) \cdot (b + U_{i+1})^2 = ahb + U_{i+1}$ for all $a, b \in U_i$. Since U_i/U_{i+1} and \mathbf{F} are isomorphic \mathcal{H} -modules with \mathcal{H} as in Lemma 4.2, we get a multiplication \cdot on \mathbf{F} with $a\varphi_b = a \cdot b^2$ for all $a, b \in \mathbf{F}$. Hence $\mathbf{F} \cong \mathbf{F}_{q_i}$, and we have $\varphi_{a \cdot b} = \varphi_a \varphi_b$. Since $a\mu_e = -a\mu_a\mu_e = -ah_a^{-1}$, we get $a\mu_e + U_{n+1} = -ah_a^{-1} + U_{n+1} = -(a + U_{n+1})^{-1}$ for all $a \in V^*$.

If $x, y, z \in V$ with $(x + U_{n+1}) \cdot (y + U_{n+1}) = z + U_{n+1}$, then $h_x h_y h_z^{-1} \in C_{\tilde{\mathcal{H}}}(V/U_{n+1}) \leq C_{\tilde{\mathcal{H}}}(U_{n+1}/U_{n+2})$ by Lemma 4.2(i). It follows that $\psi_{a \cdot b} = \psi_a \psi_b$ for all $a, b \in \mathbf{F}$.

(iii) The map ψ is multiplicative by Part (ii). Moreover the map $f(a, b) = \psi_{a+b} - \psi_a - \psi_b$ is biadditive by Lemma 3.6. For any $n \in \mathbf{Z}$, we also have that $\psi_{n \cdot e} = n^2 \psi_e$ by Proposition 4.6(6) from [DMS08b]. The claim follows.

(iv) By Lemma 4.1, the elementary abelian group \mathbf{E} carries a multiplication \cdot that turns it into a field (isomorphic to \mathbf{F}_{q_0} or \mathbf{F}_{q_1}). We let \mathbf{E}' be the subring of $\text{End}(\mathbf{E})$ defined by $\mathbf{E}' = \{R_y \mid y \in \mathbf{E}\}$, where $R_y \in \text{End}(\mathbf{E})$ denotes the right-multiplication by y . Thus \mathbf{E}' is a field which is isomorphic to \mathbf{E} .

We claim that $\psi: \mathbf{F} \rightarrow \text{End}(\mathbf{E})$ takes values in \mathbf{E}' , unless \mathbf{E} is isomorphic to \mathbf{F}_9 .

To establish the claim, we consider the little projective group of the local Moufang set at x_{n+1} , which we denote by \overline{G} . Hence $\overline{G} \cong \text{PSL}_2(\mathbf{E})$. For each $a \in \mathbf{F}$, we have $h_a \in \tilde{\mathcal{H}} \leq \mathfrak{H}_i$, with the notation of Lemma 4.2. The group \mathfrak{H}_i is contained in the Hua group \mathfrak{H} , and therefore normalises both U_∞ and U_0 ; moreover \mathfrak{H}_i fixes the apartment Σ pointwise. This implies that the image of h_a in $\text{Sym}(x_{n+1}^\perp)$ normalises a pair of opposite local roots groups at x_{n+1} . Therefore h_a is contained in the normaliser $\Gamma = N_{\text{Sym}(x_{n+1}^\perp)}(\overline{G})$, which is isomorphic to $\text{P}\Gamma\text{L}_2(\mathbf{E})$. In fact we see moreover that h_a normalises the Hua subgroup \overline{H} of \overline{G} in Γ .

Keeping the notation of Lemma 4.2, we see from that lemma that $\tilde{\mathcal{H}}/\tilde{\mathcal{H}}_0$ is isomorphic to the Hua group of the local Moufang set at x_i , and is cyclic. Moreover Lemma 4.2 ensures that $\tilde{\mathcal{H}}_0$ acts trivially on U_{n+1}/U_{n+2} , which implies that $\tilde{\mathcal{H}}_0$ acts trivially on x_{n+1}^\perp . Therefore, the image of $\tilde{\mathcal{H}}$ in $\text{Sym}(x_{n+1}^\perp)$, which we denote by Y , is cyclic.

We conclude that Y is a cyclic subgroup of $N_\Gamma(\overline{H}) \cong \mathbf{E} \rtimes \text{Aut}(\mathbf{E})$. We now invoke Proposition 3.7, which ensures that the respective images of $\tilde{\mathcal{H}}$ and \mathfrak{H}_i in $\text{Sym}(x_{n+1}^\perp)$ coincide. Since the group \mathfrak{H}_i and \mathfrak{H}_{n+1} normalize each other by Lemma 3.2, we infer that the subgroups Y and $(\mathbf{E}^*)^2$ normalize each other. Since Y is abelian, it follows that Y centralises every element of Γ which is a commutator of an element of Y and an element of $(\mathbf{E}^*)^2$. In particular, any $h \in \tilde{\mathcal{H}}$ induces a Galois automorphism σ of \mathbf{E} such that σ fixes $y^{-1}y^\sigma$ for all $y \in (\mathbf{E}^*)^2$. By Lemma 2.7 it follows that σ is the identity unless $\mathbf{E} \cong \mathbf{F}_9$. This means precisely that if $\mathbf{E} \not\cong \mathbf{F}_9$, then every element $h \in \tilde{\mathcal{H}}$ acts on \mathbf{E} as an element of \mathbf{E}' . The claim stands proven.

Assume now that ψ does not take all its values in \mathbf{E}' . Then $\mathbf{E} \cong \mathbf{F}_9$ by the claim. We then invoke Lemma 2.10, whose hypotheses are satisfied in view of Part (iii) and Lemma 3.2. This implies that $\mathbf{F} \cong \mathbf{F}_9$, and that $\{\psi_a \mid a \in \mathbf{F}\}$ generates a field $\mathbf{K} \subset \text{End}(\mathbf{E})$ isomorphic to \mathbf{F}_9 . Let $1_{\mathbf{E}}$ denote the unit element of the field \mathbf{E} , and consider the map

$$f: \mathbf{K} \rightarrow \mathbf{E} : \alpha \mapsto 1_{\mathbf{E}}\alpha.$$

The image $f(\mathbf{K})$ is an additive subgroup of \mathbf{E} containing $1_{\mathbf{E}}$. The group of invertible endomorphisms of \mathbf{E} that preserve the cyclic subgroup $\langle 1_{\mathbf{E}} \rangle$ is of order 12. It follows that the multiplicative group \mathbf{K}^* , which has order 8, contains an element that does not preserve $\langle 1_{\mathbf{E}} \rangle$. Therefore $f(\mathbf{K})$ is not contained in $\langle 1_{\mathbf{E}} \rangle$, so that f is surjective. Hence f is bijective. Notice moreover that f induces an isomorphism of additive groups from \mathbf{K} to \mathbf{E} . We now endow \mathbf{E} with a new multiplication, which is the unique multiplication that makes f an isomorphism of fields. With respect to this new multiplication, which will again be denoted by the same symbol \cdot , we see that the subring $\mathbf{E}'' = \{R_y \mid y \in \mathbf{E}\} \subset \text{End}(\mathbf{E})$ coincides with \mathbf{K} . Moreover, the multiplicative quadratic map ψ takes values in \mathbf{E}'' by construction.

At this point, we conclude that there are only two possibilities: either the multiplicative quadratic map ψ takes values in the field \mathbf{E}' , which is isomorphic to \mathbf{E} , or in the field \mathbf{E}'' , which is also isomorphic to \mathbf{E} . Moreover, we have canonical isomorphisms $\mathbf{E}' \rightarrow \mathbf{E}$ and $\mathbf{E}'' \rightarrow \mathbf{E}$ given by the map $R_y \mapsto y$. We shall now treat both cases simultaneously, by identifying \mathbf{E}' (resp. \mathbf{E}'') with \mathbf{E} via that map. With this identification, we obtain in either case that $x\psi_y = x \cdot \psi_y$ for all $x \in \mathbf{E}$ and $y \in \mathbf{F}$, where \cdot denote the multiplication of \mathbf{E} , as constructed above in the two different cases respectively.

We now apply Theorem 2.8 to the multiplicative quadratic map $\psi: \mathbf{F} \rightarrow \mathbf{E}$. This yields an embedding $\iota: \mathbf{F} \rightarrow \overline{\mathbf{E}}$ and two natural numbers $1 \leq k, l \leq r_i$ such that ψ_x coincides with the right multiplication by $\iota(x^{p^k+p^l})$ for all $x \in \mathbf{F}$. (We recall from (ii) that \mathbf{F} has order $q_i = p^{r_i}$.) If $l - k \leq \frac{r_i}{2}$, we replace ι by $\iota': \mathbf{F} \rightarrow \overline{\mathbf{E}} : a \mapsto \iota(a^{p^{r_i-k}})$. If $l - k > \frac{r_i}{2}$, we replace ι by $\iota': \mathbf{F} \rightarrow \overline{\mathbf{E}} : a \mapsto \iota(a^{p^{r_i-l}})$. In both cases we conclude that ψ_x coincides with the right multiplication by $\iota(x)^{1+p^s}$ with $0 \leq s \leq \frac{r_i}{2}$. \square

Proposition 4.4. *Assume that T is locally finite and that U is abelian of exponent p for some prime p . Let X be a finite solvable p' -subgroup of the Hua group \mathfrak{H} , and let $i \in \{0, 1\}$. Then there is a descending chain of subgroups $U_i = V_i \geq V_{i+1} \geq \dots$ satisfying the following conditions for all $n \geq i$.*

- (i) $V_n = V_{n+1} + U_{n+1}$.
- (ii) $U_{n+1} \cap V_{n+1} = U_{n+2}$.
- (iii) There is $e_n \in V_n \setminus U_{n+1}$ such that $a\mu_{e_n} \in V_n$ for all $a \in V_n^* := V_n \setminus U_{n+1}$.

(iv) V_n is invariant under X and $H_n = \langle \mu_a \mu_b \mid a, b \in V_n^* \rangle$.

Proof. We work by induction on n . For the base case, we remark that $V_i = U_i$ satisfies trivially (i) and (ii), while (iii) and (iv) are ensured by Lemma 3.1.

Assume now that we have constructed $V_i \geq \dots \geq V_n$ for $n \geq i$ with the requested properties. Set $\overline{X} = X/C_X(V_n/U_{n+2})$, $\mathcal{H} = H_n/C_{H_n}(V_n/U_{n+2})$ and $\mathcal{H}_0 = C_{\mathcal{H}}(V_n/U_{n+1})$. Since X normalizes V_n , it follows that \overline{X} , regarded as a subgroup of $\text{Aut}(V_n/U_{n+2})$, normalizes \mathcal{H} . We form the semi-direct product $\mathcal{H} \rtimes \overline{X}$.

Applying Lemma 4.2 with $V = V_n$, we see that \mathcal{H}_0 is a normal elementary-abelian p -group of \mathcal{H} and $\mathcal{H}/\mathcal{H}_0$ is cyclic of order $q_i - 1$ if $p = 2$ (resp. $\frac{q_i-1}{2}$ if p is odd). Therefore $\mathcal{H} \rtimes \overline{X}$ is a finite solvable group. By Hall's Theorem, the p' -subgroup X is contained in a Hall p' -subgroup, which is a complement for \mathcal{H}_0 in $\mathcal{H} \rtimes \overline{X}$. This implies that there is a complement \overline{H} of \mathcal{H}_0 in \mathcal{H} which is normalised by \overline{X} . Let H be the preimage of \overline{H} in H_n .

Since $\overline{H}\overline{X}$ is a p' -subgroup of $\text{Aut}(V_n/U_{n+2})$ which stabilizes U_{n+1}/U_{n+2} , we deduce from Maschke's theorem that there is an HX -invariant subgroup V_{n+1} of V_n with $V_n = V_{n+1} + U_{n+1}$ and $U_{n+1} \cap V_{n+1} = U_{n+2}$.

It remains to check the property (iii). Indeed, if we find $e \in V_{n+1} \setminus U_{n+2}$ satisfying $b\tau \in V_{n+1} \setminus U_{n+2}$ for all $b \in V_{n+1} \setminus U_{n+2}$, where $\tau = \mu_e$, then we also have $bh_c = ((b\tau + c)\tau - c\tau)\tau + c \in V_{n+1} \setminus U_{n+2}$ for all $b, c \in V_{n+1} \setminus U_{n+2}$ (see Definition 3.4 in [DMS09]). Hence (iv) follows from (iii).

Since all the complements of \mathcal{H}_0 in \mathcal{H} are conjugate, we deduce from Lemma 4.2(iv) there exist elements $a, e \in V_{n+1} \setminus U_{n+2}$ and $x, y \in U_{n+1}$ such that \overline{H} is generated by the image of $h = \mu_{e+x}\mu_{a+y}$. Those elements a, e, x, y are fixed for the rest of the proof. Moreover, we set $\tau = \mu_e$.

Given $b \in V_n$, we set $\bar{b} = b + U_{n+2}$. Moreover, we set $\mathbf{F} = V_{n+1}/U_{n+2}$. Since \mathbf{F} and V_n/U_{n+1} are isomorphic H -modules, by Lemma 4.3(ii), there is a multiplication \cdot on \mathbf{F} such that $(\mathbf{F}, +, \cdot)$ is a field with neutral element \bar{e} , $-\bar{b}^{-1} \equiv \overline{b\tau} \pmod{U_{n+1}/U_{n+2}}$ and $\overline{bh_c} \equiv \bar{b} \cdot \bar{c}^2 \pmod{U_{n+1}/U_{n+2}}$ for all $b, c \in V_{n+1}, b \notin U_{n+1}$.

Since H_n normalizes V_n we have $b\tau = -b\mu_b\tau = -bh_b^{-1} \in V_n \setminus U_{n+1}$ for $b \in V_n \setminus U_{n+1}$. Therefore by Lemma 3.4 the map τ induces a permutation on $(V_n/U_{n+2}) \setminus (U_{n+1}/U_{n+2})$ which we will also call τ .

We will prove the following claim:

$$\bar{b}\tau = -\bar{b}^{-1} \quad \text{for all } b \in V_{n+1} \setminus U_{n+1}.$$

As observed above, this claim implies that V_{n+1} is invariant under H_{n+1} , which completes the proof of the proposition.

Since h and $h_a = \mu_e\mu_a$ induce the same map on V_n/U_{n+1} , the element \bar{a} generates the multiplicative group of \mathbf{F} . Since $U_i/U_{i+1}, V_n/U_{n+1}$ and \mathbf{F} are isomorphic H -modules, H has at most two orbits on $\mathbf{F} \setminus \{\bar{0}\}$ (namely the squares and the non-squares) with representatives \bar{e} and \bar{a} . Therefore, in order to compute $\bar{b}\tau$ for all $\bar{b} \in \mathbf{F}^\#$, it suffices to compute $e h^j \tau$ and $a h^j \tau$ for all $j \geq 0$.

We have the following sequence of congruences modulo U_{n+2} :

$$\begin{aligned}
eh^j\tau &\equiv e\mu_{e+x}^2h^j\mu_{e+x}^2\tau \\
&\equiv (e+x-x)\tau^2\mu_{e+x}h^{-j}\mu_{e+x}\tau && \text{because } \mu_{e+x} \text{ inverts } h \\
&\equiv ((e+x)\tau - xh_{e+x}^{-1})h_{e+x}h^{-j}h_{e+x}^{-1} && \text{by Lemma 3.4} \\
&\equiv (-(e+x)-x)h^{-j}h_{e+x}^{-1} && \text{by Lemma 2.1(iv)} \\
&\equiv (-e-2x)h^{-j}h_{-e+x} && \text{because } (e+x)\tau \equiv -e+x \pmod{U_{n+2}} \text{ by Lemma 3.4} \\
&\equiv -eh^{-j}h_{e-x} - 2xh^{-j} && \text{by Proposition 3.7 and Lemma 2.1(i)} \\
&\equiv -eh^{-j} + xh_{-eh^{-j},e\tau}h_e - 2xh^{-j} && \text{by Lemma 3.6 and because } eh^{-j}h_x \in U_{n+2} \\
&\equiv -eh^{-j} - 2xh^{-j} + xh_{eh^{-j},e}.
\end{aligned}$$

Set $\mathbf{E} = U_{n+1}/U_{n+2}$. By Lemma 4.3(iv), there is a multiplication \cdot on \mathbf{E} such that $(\mathbf{E}, +, \cdot)$ is a field and such that there is a natural number $0 \leq s \leq \frac{r_i}{2}$ and an embedding ι from \mathbf{F} to the algebraic closure of \mathbf{E} satisfying $\overline{ch_b} = \overline{c} \cdot \iota(\overline{b})^{1+r}$ for $r = p^s$. Since $\overline{eh^j} = \overline{a}^{2j}$, we deduce from the computation above that

$$\overline{a}^{2j}\tau = -\overline{a}^{-2j} - 2\overline{x}\iota(\overline{a})^{-2j(1+r)} + \overline{x}\iota(\overline{a}^{-2jr} + \overline{a}^{-2j}).$$

We now compute a similar formula for the non-squares of \mathbf{F} . Notice that $\mu_{a+y}h^j\mu_{e+x} = h^{-1-j}$. Therefore, by a similar argument, we obtain the following sequence of congruences modulo U_{n+2} :

$$\begin{aligned}
ah^j\tau &\equiv (a+y-y)\mu_{a+y}^2h^j\mu_{e+x}^2\tau \\
&\equiv (a+y-y)\tau h_{a+y}h^{-j-1}\mu_{e+x}\tau && \text{since } \mu_{a+y}h^j\mu_{e+x} = h^{-1-j} \\
&\equiv ((a+y)\tau - yh_{a+y}^{-1})h_{a+y}h^{-j-1}h_{e+x}^{-1} && \text{by Lemma 3.4} \\
&\equiv ((a+y)\mu_{a+y} - y)h^{-j-1}h_{e+x}^{-1} \\
&\equiv (-a-y-y)h^{-j-1}h_{e+x}^{-1} && \text{by Lemma 2.1(iv)} \\
&\equiv -ah^{-j-1}h_{e+x}^{-1} - 2yh^{-j-1} && \text{by Proposition 3.7} \\
&\equiv -ah^{-j-1}h_{-e+x} - 2yh^{-j-1} && \text{because } (e+x)\tau \equiv -e+x \pmod{U_{n+2}} \text{ by 3.4} \\
&\equiv -ah^{-j-1}h_{e-x} - 2yh^{-j-1} && \text{by Lemma 2.1(i)} \\
&\equiv -ah^{-j-1} + xh_{-ah^{-j-1},e\tau}h_e - 2yh^{-j-1} && \text{by Lemma 3.6 and because } eh^{-j}h_x \in U_{n+2} \\
&\equiv -ah^{-j-1} + xh_{-ah^{-j-1},-e} - 2yh^{-j-1}
\end{aligned}$$

Since $\overline{ah^j} = \overline{a}^{2j+1}$, this implies

$$\overline{a}^{2j+1}\tau = -\overline{a}^{-2j-1} - 2\overline{z}\iota(\overline{a})^{-(2j+1)\frac{1+r}{2}} + \overline{x}\iota(\overline{a}^{-(2j+1)r} + \overline{a}^{-(2j+1)})$$

where $\overline{z} = \overline{y}\iota(\overline{a})^{-\frac{1+r}{2}}$.

Combining the two conclusions of those two computations, we deduce that for any $\overline{b} \in \mathbf{F} \setminus \{\overline{0}\}$, we have

$$\overline{b}\tau = -\overline{b}^{-1} - 2\overline{z}_{\overline{b}}\iota(\overline{b})^{-\frac{1+r}{2}} + \overline{x}\iota(\overline{b}^{-r} + \overline{b}^{-1}), \quad (*)$$

where $\overline{z}_{\overline{b}} = \overline{x}$ if \overline{b} is a square in \mathbf{F} , and $\overline{z}_{\overline{b}} = \overline{z}$ if \overline{b} is a non-square in \mathbf{F} .

Since $(b+c)\tau \equiv b\tau + ch_b^{-1} \pmod{U_{n+2}}$ for $b \in V_{n+1}^*$ and $c \in U_{n+1}$ by Lemma 3.4, we have

$$(\bar{b} + \bar{c})\tau = -\bar{b}^{-1} - 2z_b\iota(\bar{b})^{-\frac{1+r}{2}} + \bar{x}\iota(\bar{b}^{-1} + \bar{b}^{-r}) + \bar{c}\iota(\bar{b})^{-1-r}.$$

By [DMS09, Proposition 4.1.1], we have $\tau = \mu_e = \alpha_e \alpha_{-e\tau^{-1}}^\tau \alpha_{-(-e\tau^{-1})\tau} = \alpha_e \alpha_e^\tau \alpha_e$. In particular, for $b \in V_{n+1}^*$, we have $\bar{b}\tau = ((\bar{b} + \bar{c})\tau + \bar{c})\tau + \bar{c}$. Therefore, for any $b \in V_{n+1}$ with $\bar{b} \neq -\bar{c}, \bar{0}$, we obtain:

$$\begin{aligned} & -\bar{b}^{-1} - 2z_{\bar{b}}\iota(\bar{b})^{-\frac{1+r}{2}} + \bar{x}(\iota(\bar{b})^{-r} + \iota(\bar{b})^{-1}) \\ &= \bar{b}\tau \\ &= ((\bar{b} + \bar{c})\tau + \bar{c})\tau + \bar{c} \\ &= (-\bar{b} + \bar{c})^{-1} - 2\bar{z}_{\bar{b}+\bar{c}}\iota(\bar{b} + \bar{c})^{-\frac{1+r}{2}} + \bar{x}(\iota(\bar{b} + \bar{c})^{-r} + \iota(\bar{b} + \bar{c})^{-1}) + \bar{c})\tau + \bar{c} \\ &= (\bar{b}(\bar{b} + \bar{c})^{-1} - 2\bar{z}_{\bar{b}+\bar{c}}\iota(\bar{b} + \bar{c})^{-\frac{1+r}{2}} + \bar{x}(\iota(\bar{b} + \bar{c})^{-r} + \iota(\bar{b} + \bar{c})^{-1}))\tau + \bar{c} \\ &= -\bar{b}^{-1}(\bar{b} + \bar{c}) - 2\bar{z}_{\bar{b}+\bar{c}}\iota(\bar{b} + \bar{c})^{-\frac{1+r}{2}}\iota(\bar{b})^{-1-r}\iota(\bar{b} + \bar{c})^{1+r} \\ &\quad + \bar{x}(\iota((\bar{b} + \bar{c})^{-r} + (\bar{b} + \bar{c})^{-1}))\iota(\bar{b})^{-1-r}\iota(\bar{b} + \bar{c})^{1+r} \\ &\quad - 2\bar{z}_{\bar{b}(\bar{b}+\bar{c})}\iota(\bar{b})^{-\frac{1+r}{2}}\iota(\bar{b} + \bar{c})^{\frac{1+r}{2}} + \bar{x}\iota(\bar{b}^{-r}((\bar{b} + \bar{c})^r + \bar{b}^{r-1}(\bar{b} + \bar{c}))) + \bar{c} \\ &= -\bar{b}^{-1} - 2\bar{z}_{\bar{b}+\bar{c}}\iota(\bar{b} + \bar{c})^{\frac{1+r}{2}}\iota(\bar{b})^{-1-r} + \bar{x}\iota(2\bar{b}^{1+r} + 2\bar{b} + 2\bar{c} + 2\bar{b}^r)\iota(\bar{b})^{-1-r} \\ &\quad - 2\bar{z}_{\bar{b}(\bar{b}+\bar{c})}\iota(\bar{b})^{-\frac{1+r}{2}}\iota(\bar{b} + \bar{c})^{\frac{1+r}{2}}. \end{aligned}$$

It follows that

$$\begin{aligned} f(\bar{b}) &= \bar{x}\iota(2\bar{b}^{1+r} + \bar{b}^r + \bar{b} + 2\bar{c}) + 2\bar{z}_{\bar{b}}\iota(\bar{b})^{\frac{1+r}{2}} - 2\bar{z}_{\bar{b}+\bar{c}}\iota(\bar{b} + \bar{c})^{\frac{1+r}{2}} - 2\bar{z}_{\bar{b}(\bar{b}+\bar{c})}\iota(\bar{b})^{\frac{1+r}{2}}\iota(\bar{b} + \bar{c})^{\frac{1+r}{2}} \\ &= \bar{0} \end{aligned}$$

for all $\bar{b} \in \mathbf{F} \setminus \{-\bar{c}, \bar{0}\}$.

Assume that $\text{char}(\mathbf{F}) = 2$. If $\bar{b} = \bar{c} = -\bar{c}$, then $\bar{b}\tau = -\bar{b}^{-1}$ by (*). Otherwise we have $\bar{b} \in \mathbf{F} \setminus \{-\bar{c}, \bar{0}\}$, and we get $\bar{x}(\iota(\bar{b}) + \iota(\bar{b})^r) = \bar{0}$ from the equation above, so that $r = 1$ or $\bar{x} = \bar{0}$. In both cases we have $\bar{b}\tau = -\bar{b}^{-1}$ by (*), and the desired claim follows.

We assume henceforth that $\text{char}(\mathbf{F}) \neq 2$. We shall prove that $\bar{x} = \bar{z}$, and that the common value is $\bar{0}$ unless $r = 1$. In either case, the desired equation $\bar{b}\tau = -\bar{b}^{-1}$ follows readily from (*), which finishes the proof.

As a first case, assume that $r = 1$. If $\mathbf{F} = \mathbf{F}_3$, we apply the equation $f(\bar{b}) = \bar{0}$ with $\bar{b} = \bar{c}$; it yields $\bar{x} = \bar{z}$, as claimed. Otherwise, there exists a non-square \bar{b} in $\mathbf{F} \setminus \{-\bar{c}, \bar{0}\}$. In that case, the equation $f(\bar{b}) = \bar{0}$ yields

$$-2\bar{z}_{\bar{c}+\bar{b}}\iota(\bar{c} + \bar{b}) + \bar{x}\iota(2\bar{c} + 2\bar{b} + 2\bar{b}^2) - 2\bar{z}_{\bar{b}(\bar{c}+\bar{b})}\iota(\bar{b}^2 + \bar{b}) + 2\bar{z}\iota(\bar{b}) = \bar{0}.$$

Therefore, if $\bar{b} + \bar{c}$ is a square, we get $2\bar{x}\iota(\bar{b})^2 - 2\bar{z}\iota(\bar{b})^2 = 0$, and thus $\bar{x} = \bar{z}$, as claimed. Similarly, if $\bar{b} + \bar{c}$ is a non-square, we get $2\bar{x}\iota(\bar{c} + \bar{b}) - 2\bar{z}\iota(\bar{c} + \bar{b}) = \bar{0}$, and again $\bar{x} = \bar{z}$, as claimed.

We now consider the remaining case $r \geq 3$. The goal is then to show that $\bar{x} = \bar{z} = \bar{0}$.

Suppose for a contradiction that $\bar{x} \neq \bar{0}$. Let $1 = \iota(\bar{c})$ denote the unit element of \mathbf{E} , and Y be an indeterminate. We define four polynomials $f_1(Y), \dots, f_4(Y) \in \mathbf{E}[Y]$ as follows.

The polynomial $f_1(Y)$ is obtained by replacing $\iota(\bar{b})$ by Y , and $\bar{z}_{\bar{b}}, \bar{z}_{\bar{e}+\bar{b}}$ and $\bar{z}_{\bar{b}(\bar{e}+\bar{b})}$ by \bar{x} . The polynomial $f_2(Y)$ is obtained by replacing $\bar{z}_{\bar{b}}$ by \bar{x} , and $\bar{z}_{\bar{e}+\bar{b}}$ and $\bar{z}_{\bar{b}(\bar{e}+\bar{b})}$ by \bar{z} . The polynomial $f_3(Y)$ is obtained by replacing $\bar{z}_{\bar{b}}$ and $\bar{z}_{\bar{b}(\bar{e}+\bar{b})}$ by \bar{z} , and $\bar{z}_{\bar{e}+\bar{b}}$ by \bar{x} . The polynomial $f_4(Y)$ is obtained by replacing $\bar{z}_{\bar{b}}$ and $\bar{z}_{\bar{e}+\bar{b}}$ by \bar{z} and $\bar{z}_{\bar{b}(\bar{e}+\bar{b})}$ by \bar{x} . One computes that:

$$\begin{aligned} f_1(Y) &= \bar{x} \left(2Y^{r+1} + Y^r + Y + 2 - 2(Y^2 + Y)^{\frac{1+r}{2}} - 2(Y+1)^{\frac{1+r}{2}} + 2Y^{\frac{1+r}{2}} \right), \\ f_2(Y) &= \bar{x} \left(2Y^{r+1} + Y^r + 2Y^{\frac{1+r}{2}} + Y + 2 \right) - 2\bar{z}(Y+1)^{\frac{1+r}{2}}(Y^{\frac{1+r}{2}} + 1), \\ f_3(Y) &= \bar{x} \left(2Y^{r+1} + Y^r - 2(Y+1)^{\frac{1+r}{2}} + Y + 2 \right) - 2\bar{z}Y^{\frac{1+r}{2}} \left((Y+1)^{\frac{1+r}{2}} - 1 \right), \\ f_4(Y) &= \bar{x} \left(2Y^{r+1} + Y^r - 2(Y^2 + Y)^{\frac{1+r}{2}} + Y + 2 \right) - 2\bar{z} \left((Y+1)^{\frac{1+r}{2}} - Y^{\frac{1+r}{2}} \right). \end{aligned}$$

Observe that all four polynomials are non-zero. Indeed, the term of highest degree in $f_2(Y)$ and $f_3(Y)$ is $2(\bar{x} - \bar{z})Y^{r+1}$. Moreover, we have $f_4(\bar{0}) = 2(\bar{x} - \bar{z})$. For f_1 , we see that the term of highest degree is $\bar{x} \frac{(r+1)(r-1)}{4} Y^{r-1}$ if $r-1 > \frac{r+1}{2}$, i.e. if $r > 3$. On the other hand, if $r = 3$, then $\text{char}(\mathbf{F}) = p = 3$ since r is a power of p by definition. In that case we have $f_1(Y) = \bar{x}Y^2$. This confirms that all four polynomials are non-zero.

Since $f(\bar{b}) = \bar{0}$ for all $\bar{b} \in \mathbf{F} \setminus \{-\bar{e}, \bar{0}\}$ by the above, we deduce that the polynomial

$$P(Y) = f_1(Y)f_2(Y)f_3(Y)f_4(Y)(Y^2 + Y) \in \mathbf{E}[Y]$$

vanishes on the field $\iota(\mathbf{F})$. Since f_1 and f_4 have degree $\leq r-1$ while f_2 and f_3 have degree $r+1$, we see that P has degree $\leq 4r+2$. Therefore $|\mathbf{F}| = q_i \leq 4r+2 < 5r \leq 5\sqrt{q_i}$. This implies that $q_i = 9$ and $r = 3$, so that $f_1(Y) = \bar{x}Y^2$. For $\bar{b} = \bar{e}$ both \bar{b} and $\bar{e} + \bar{b}$ are squares. We must then have $f_1(\bar{b}) = \bar{0}$, which yields $\bar{x}\iota(\bar{b})^2 = \bar{0}$, a contradiction. This proves that $\bar{x} = \bar{0}$.

Assume now that $\bar{z} \neq \bar{0}$. We now consider the polynomial $Q(Y) = f_3(Y)f_4(Y)$, so that $Q(\iota(\bar{b})) = \bar{0}$ for all non-squares $\bar{b} \in \mathbf{F} \setminus \{-\bar{e}, \bar{0}\}$. Now Q has degree $r+1 + \frac{r+1}{2} - 1$, hence $\frac{3}{2}r + \frac{1}{2} \geq \frac{1}{2}(q_i - 1)$. We deduce that $q_i \leq 3r+2 < 4r \leq 4\sqrt{q_i}$, and as before it follows that $q_i = 9$ and $r = 3$. In that case, one computes $Q(Y) = -\bar{z}Y^3(Y-1)^2$. This implies that every non-square in \mathbf{F} must be zero, which is absurd. Hence $\bar{z} = \bar{0}$, and the proof is complete. \square

Lemma 4.5. *Assume that the root group U_∞ is a closed subgroup of $\text{Aut}(T)$. Let $V, W \leq U$ subgroups such that $U_0 = U_1 + V$ and $U_1 = U_2 + W$. Then the centraliser $C_{\mathfrak{H}}(V+W)$ is trivial.*

Proof. Let $h \in C_{\mathfrak{H}}(V+W)$. Then $C_U(h)$ is a root subgroup of U , see Lemma 6.2.3 from [DMS09]. Let $a \in V \setminus U_1$ and $b \in W \setminus U_2$ and set $t = \mu_a \mu_b$. Then t conjugates U_n to U_{n+2} for all $n \in \mathbf{Z}$. Since $a, b \in C_U(h)$ and since $C_U(h)$ is a root subgroup, it follows that t normalizes $C_U(h)$. Therefore the set

$$\left\{ \sum_{i=n}^m a_i t^i + b_i t^i \mid a_i \in V, b_i \in W; n, m \in \mathbf{Z}, n < m \right\}$$

is contained in $C_U(h)$. Identifying U with U_∞ via α , we may view U as a topological group and infer that $C_U(h)$ is dense in U . Since h induces a continuous automorphism on U , it follows that $C_U(h)$ is closed, so that $C_U(h) = U$ and $h = 1$. \square

Proposition 4.6. *Assume that T is locally finite and that U is abelian of exponent p for some prime p . Assume further that the root group U_∞ is closed in $\text{Aut}(T)$.*

For any finite solvable p' -subgroup $X \leq \mathfrak{H}$ and any $i \in \{0, 1\}$, there is a finite subgroup $V \leq U_i$ satisfying the following conditions:

- (i) V is a root subgroup.
- (ii) U_i is the direct sum of V and U_{i+1} .
- (iii) The group $\tilde{\mathcal{H}} = \langle \mu_a \mu_b \mid a, b \in V^\# \rangle$ is finite cyclic of order prime to p .
- (iv) V is normalised by X and $\tilde{\mathcal{H}}$.

Proof. Let $U_i = V_0 \geq V_1 \geq V_2 \geq \dots$ be a descending chain afforded by applying Proposition 4.4. Remark that U_n is an open subgroup of U for all n . Since $V_n \geq U_{n+1}$ for all n , we infer that V_n is also open, hence closed, for all n .

Set $V = \bigcap_{n \geq 0} V_n$. For every $a \in U_i \setminus U_{i+1}$ and $n \geq 0$, we have $U_{i+1}a \cap V_n \neq \emptyset$. By compactness, it follows that $V \cap U_{i+1}a = \bigcap_{n \in \mathbf{N}} (V_n \cap U_{i+1}a)$ is non-empty. Therefore $U_i = V + U_{i+1}$. Moreover, we have $V_n \cap U_{i+1} = U_{n+1}$ for all n , so that $V \cap U_{i+1} = \bigcap_{n \in \mathbf{N}} U_n = 0$. Thus V is a complement for U_{i+1} in U_i , thereby proving (ii). Moreover, since we have $\tilde{\mathcal{H}} \leq N_{\mathfrak{H}}(V_n)$ for all $n \geq 0$, we see that $\tilde{\mathcal{H}} \leq N_{\mathfrak{H}}(V)$. Since X normalises V_n for all n , we also obtain (iv).

Let $e_n \in V_n$ be the element afforded by Proposition 4.4(iii); let $e \in V$ be any accumulation point of the sequence (e_n) . Then the element $\tau = \mu_e$ is an accumulation point of the sequence μ_{e_n} by [CDM13, Proposition 3.8]. The fact that $a\tau \in V$ for all $a \in V^\#$ now follows from Proposition 4.4(iii), in view of the definition of V .

Remark that the Moufang subset (V, μ_a) with $a \in V^\#$ is isomorphic to the local Moufang set at x_i . It is thus isomorphic to $\mathbb{M}(\mathbf{F}_{q_i})$ by Lemma 4.1, we deduce from Lemma 2.6 that $\tilde{\mathcal{H}}$ is cyclic of order $q_i - 1$ if $p = 2$ (resp. $\frac{q_i - 1}{2}$ if p is odd). \square

4.3 Construction of a twin tree lattice

RGD-systems and their relations with twin tree lattices have been reviewed in Section 2.3. The main step in our proof of Theorem 1.1 is given by the following.

Theorem 4.7. *Let T be a locally finite tree and p be a prime. Let $(U_\epsilon \mid \epsilon \in \partial T)$ be a collection of closed, abelian subgroups of exponent p in $\text{Aut}(T)$ such that $(\partial T, (U_\epsilon \mid \epsilon \in \partial T))$ is a Moufang set. Let $G = \overline{\langle U_\epsilon \mid \epsilon \in \partial T \rangle}$ be the closure of its little projective group.*

Then, for any pair of distinct ends $+\epsilon, -\epsilon \in \partial T$, there is a dense subgroup $\Gamma \leq G$, a finite subgroup X of the Hua group, and a collection of finite root subgroups $\{U_n^\epsilon \leq U_{\epsilon\epsilon}\}_{n \in \mathbf{Z}, \epsilon \in \{+, -\}}$ such that $(\Gamma, (U_n^\epsilon)_{n \in \mathbf{Z}, \epsilon \in \{+, -\}}, X)$ is an RGD-system.

Proof. Let $V^0 \leq U_0$ be the root subgroup afforded by applying Proposition 4.6 with $i = 0$ and $X = 1$. Set $Y_0 = \langle \mu_a \mu_b \mid a, b \in V^\# \rangle$. Then Y_0 is a finite solvable p' -subgroup of \mathfrak{H} . We then apply Proposition 4.6 with $X = Y_0$ and $i = 1$. We obtain a Y_0 -invariant root subgroup W^0 of U_1 which is a complement for U_2 in U_1 . Set $Z_0 = \langle \mu_a \mu_b \mid a, b \in (W^0)^\# \rangle$. Since Y_0 normalizes W^0 , it also normalizes Z_0 . Since U_1/U_2 and W^0 are isomorphic Y_0 -modules and since $\mathfrak{H}/C_{\mathfrak{H}}(U_1/U_2)$ is a p' -group, we see that $X_0 = Y_0 Z_0$ is a finite solvable p' -group.

We now apply Proposition 4.6 once more, with $X = X_0$ and $i = 1$. This yields a X_0 -invariant root subgroup V^1 which is a complement for U_1 in U_0 . Set $Y_1 = \langle \mu_a \mu_b \mid a, b \in$

$(V^1)^\#$). As above we conclude that Y_1 is normalized by X_0 . Set $X_1 = Y_1 X_0$. We repeat this process and get an ascending series of finite solvable p' -groups $X_0 \leq X_1 \leq X_2 \leq X_3 \leq \dots$ such that X_i normalizes a complement V^i of U_1 in U_0 and a complement W^i of U_2 in U_1 , such that V^i and W^i are both root subgroups. Since X_i acts faithfully on $V^i + W^i$ by Lemma 4.5, this chain must stabilize and the process terminates. We therefore obtain a finite p' -group X which normalizes a complement V for U_1 in U_0 and a complement W for U_2 in U_1 . Moreover V and W are root subgroups, and the sets $\{\mu_a \mu_b \mid a, b \in V^*\}$ and $\{\mu_a \mu_b \mid a, b \in W^*\}$ are contained in X . Notice that if $a \in V^* \cup W^*$ and $x \in X$, then $x^{\mu_a} = \mu_a x \mu_a = x \mu_{ax} \mu_a \in X$.

Now we fix $a \in V^*$, $b \in W^*$ and set $s_0 = \mu_a$, $s_1 = \mu_b$ and $t = s_0 s_1$. We define $U_0^+ = \alpha(V)$, $U_1^+ = \alpha(W)$, $U_0^- = (U_0^+)^{s_0}$, $U_{-1}^- = (U_1^+)^{s_1}$, $U_{2n+i}^+ = (U_i^+)^{t^n}$ and $U_{2n+i}^- = (U_i^-)^{t^n}$ for $n \in \mathbf{Z}$ and $i \in \{0, 1\}$. Note that t normalizes X and thus X normalizes U_n^ϵ for $n \in \mathbf{Z}$ and $\epsilon \in \{+, -\}$. Let Γ be the subgroup of G generated by X and U_n^ϵ for all $n \in \mathbf{Z}$ and $\epsilon \in \{+, -\}$. We proceed to verify that $(\Gamma, (U_n^\epsilon)_{n \in \mathbf{Z}, \epsilon \in \{+, -\}}, X)$ is an RGD-system by checking the axioms successively.

(RGD0) V and W are non-trivial and so is U_n^ϵ for all $n \in \mathbf{Z}$ and $\epsilon \in \{+, -\}$.

(RGD1) Since U is abelian, $[U_n^\epsilon, U_m^\epsilon] = 1$ for $n, m \in \mathbf{Z}$ and $\epsilon \in \{+, -\}$. Thus (RGD1) follows.

(RGD2) If $c \in U_0^+$, then $s_0 \mu_c \in X$, so μ_c induces a fundamental reflection of the root system. The same holds for U_1^- , so (RGD2) follows.

(RGD3) Let $U_+ = \langle U_n^+ \mid n \in \mathbf{Z} \rangle$. Since $U \cap U^{s_i} = 1$ for $i = 0, 1$, we have $U_n^- \cap U_+ \leq U_n^- \cap U = 1$ for all $n \in \mathbf{Z}$, so (RGD3) holds.

(RGD4) By definition the group Γ is generated by X and $(U_n^\epsilon)_{n \in \mathbf{Z}, \epsilon \in \{+, -\}}$.

This confirms that $(\Gamma, (U_n^\epsilon)_{n \in \mathbf{Z}, \epsilon \in \{+, -\}}, X)$ is an RGD-system. \square

4.4 Endgame

We now assemble all the results collected thus far to complete the proof of our main theorem.

Proof of Theorem 1.1. Let ∞, \mathfrak{o} be two different elements of ∂T . By [DMS09, Proposition 7.2.2], the root group U_∞ is either torsion-free and uniquely divisible, or of exponent p for some prime p . In the former case, the desired conclusion follows from [CDM13, Theorem B]. We assume henceforth that U_∞ is of exponent p .

By Theorem 4.7 there are subgroups $U_n^+ \leq U_\infty$ and $U_n^- \leq U_{\mathfrak{o}}$, a dense subgroup $\Gamma \leq G = \langle U_\infty \cup U_{\mathfrak{o}} \rangle$, and a subgroup $H \leq \mathfrak{H}$ such that $(\Gamma, (U_n^\epsilon)_{n \in \mathbf{Z}, \epsilon \in \{+, -\}}, H)$ is a RGD-system. Since the root groups are abelian and in view of Lemma 4.1, we see that the corresponding Moufang twin tree is a commutative SL_2 -twin tree. The desired conclusion now follows from Theorem 2.16 and Remarks 2.17. \square

5 Applications

5.1 Boundary-transitive tree automorphism groups

Proof of Corollary 1.2. Suppose that (i) holds. Let $\infty \in \partial T$ be the repelling fixed point of h . By Corollary 2.6 from [CDM13], the group $\mathrm{con}(h)$ is closed, so by Theorem C of loc. cit. T is boundary Moufang with the conjugates of $\mathrm{con}(h)$ as root groups. Let G^\dagger be the normal subgroup of G generated by the conjugates of $\mathrm{con}(h)$. By Theorem 1.1 there is a

non-Archimedean local field k , a simply connected, k -simple algebraic group \mathbf{G} defined over k and an epimorphism $\varphi : \mathbf{G}(k) \rightarrow G^\dagger$ with $\ker(\varphi) = Z(\mathbf{G}(k))$. Moreover G^\dagger is closed in $\text{Aut}(T)$. By Proposition 3.5 from [CDM13], it follows that $G^\dagger = G^{(\infty)}$. This proves that (ii) holds.

The other direction follows from the well-known properties of rank-one simple algebraic groups over local fields. \square

Proof of Corollary 1.3. By Theorem 8.1 from [CCMT], the group G is either an almost connected rank one simple Lie group, or G has a continuous proper, faithful action by automorphisms on a thick locally finite tree T , which is doubly transitive on ∂T . In the former case, the desired conclusion holds since all simple Lie groups are algebraic. In the latter case, one observes that the given element h must act as a hyperbolic automorphism of T , since otherwise its contraction group $\text{con}(h)$ would be trivial, forcing G to be compact, hence trivial. Therefore, the desired conclusion follows from Corollary 1.2. \square

5.2 Sharply 3-transitive groups on compact sets

Proof of Corollary 1.4. Assume that G is a σ -compact group acting sharply 3-transitively and continuously on a compact set Ω . By Theorem B of [CD12], the group G is a rank one simple Lie group, in which case the conclusion follows from [Tit51] (and the condition (i) is then automatically satisfied), or there is a locally finite tree T on which G acts properly, vertex-transitively and continuously by automorphism and a G -equivariant homeomorphism $f : \Omega \rightarrow \partial T$. We stick henceforth to the latter case.

Assume that (i) holds. Let $\omega \in \Omega$ and $U_\omega \leq G_\omega$ be a normal subgroup acting regularly on $\Omega \setminus \{\omega\}$. It follows that any two non-trivial elements of U_ω are conjugate in G_ω . Given an involution $\tau \in G_\omega$ and any element $u \in U_\omega$, we see that the commutator $[\tau, u]$ is a product of two involutions which belongs to U_ω . Therefore every element of U_ω is a product of two involutions. This implies that the group U_ω must be abelian (see [Ker74, Theorem 3.7] for more details). Therefore, so is its closure $\overline{U_\omega}$. Since any transitive action of an abelian group is regular, it follows that $\overline{U_\omega}$ also acts regularly on $\Omega \setminus \{\omega\}$. This implies that $U_\omega = \overline{U_\omega}$. In other words U_ω is a closed subgroup of G .

The hypotheses of Theorem 1.1 are thus satisfied, and we infer that the little projective group $G^\dagger = \langle U_\omega \mid \omega \in \Omega \rangle$ is of the form $\mathbf{G}(k)/Z$, where \mathbf{G} is a rank one algebraic group with abelian root groups over a local field k , and that T is isomorphic to the Bruhat–Tits tree of $\mathbf{G}(k)$. As recalled in the introduction, we must have either $\mathbf{G}(k) \cong \text{SL}_2(D)$, where D is a finite-dimensional central division algebra over k , or $\mathbf{G}(k) \cong \text{SU}_2(D, h)$, where D is the quaternion central division algebra over k , and h is an antihermitian sesquilinear form of Witt index 1 relative to an involution σ of the first kind such that the space of symmetric elements D^σ has dimension 3. Direct computations show that the only group $\mathbf{G}(k)$ on that list satisfying the condition that the pointwise stabiliser of any triple of distinct points of ∂T acts trivially, is the group $\mathbf{G}(k) \cong \text{SL}_2(k)$. This implies that G must be isomorphic to a subgroup of $\text{Aut}(\text{PSL}_2(k)) = \text{P}\Gamma\text{L}_2(k)$, thereby proving (ii).

That (ii) implies (i) is again clear, the regular normal subgroups being afforded by the maximal unipotent subgroups of $\text{PSL}_2(k)$. \square

We note here that $\text{P}\Gamma\text{L}_2(k)$ may contain other sharply 3-transitive subgroups than $\text{PGL}_2(k)$. For example, if σ is an involutory Galois automorphism of k and $\varphi : k^* \rightarrow \langle \sigma \rangle$ is a homomor-

phism with $\varphi(a^\sigma) = \varphi(a)$ for all $a \in k^*$, then the group

$$G = \{\varphi(\det g)g \mid g \in GL_2(k)\}/Z(GL_2(k))$$

acts sharply 3-transitive on $\mathbb{P}^1(k)$.

In general, one can define the **permutational characteristic** of a sharply 2-transitive group G on a set Ω as follows (see also [Ker74, §3]). Let J be the collection of all involutions in G , which is clearly a non-empty single conjugacy class in G . If the elements of J have no fixed point in Ω , then one sets $\text{char } G = 2$. If that is not the case, then one proves that the set J^{2*} consisting of all elements of G that are the product of two distinct involutions, forms a conjugacy class G , the order of whose elements is either infinite, or an odd prime p . One sets $\text{char } G = 0$ or $\text{char } G = p$ accordingly. If G is sharply 3-transitive on Ω , then the stabiliser G_ω of a point $\omega \in \Omega$ is sharply 2-transitive on $\Omega \setminus \{\omega\}$, and the assignment $\text{char } G = \text{char } G_\omega$ defines the permutational characteristic of G .

The results in [Ker74, §13] show that if G is sharply 3-transitive of characteristic $p = 3$ or $p \equiv 1 \pmod{3}$, then the point stabiliser G_ω contains a normal subgroup U_ω acting regularly on $\Omega \setminus \{\omega\}$. Therefore Corollary 1.4 yields the following.

Corollary 5.1. *Let G be a σ -compact locally compact group and Ω be a compact G -space, such that the G -action on Ω is sharply 3-transitive. If $\text{char } G = 3$ or $\text{char } G \equiv 1 \pmod{3}$, then there is a non-Archimedean local field k and a group H with $\text{PSL}_2(k) \leq H \leq \text{P}\Gamma\text{L}_2(k)$ such that (G, Ω) and $(H, \mathbb{P}^1(k))$ are isomorphic permutation groups.*

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